

Models of Mechanical Systems: Lagrangian Variables

For (sub)system models of phenomena in the domain of mechanical translation or rotation, neither power variables nor energy variables are necessarily the best choice. Mechanical inertias have linear constitutive equations (except at speeds approaching light speed — a rare occurrence for systems of interest to a practicing engineer). Thus it is convenient to work with flows (speeds) of independent inertias as state variables. On the other hand, mechanical capacitors — elastic elements — frequently have nonlinear constitutive equations. Even in the linear case, rather than work with efforts, it is generally easier to identify displacements and relate their time derivatives to the speeds of associated inertias. For mechanical systems the most convenient state variables are the flows of independent inertias and the corresponding displacements — which may be termed Lagrangian state variables. An obvious hybrid of the above two procedures can be used to derive state equations. It is illustrated by the following simple example.

Example: Linear Mechanical System

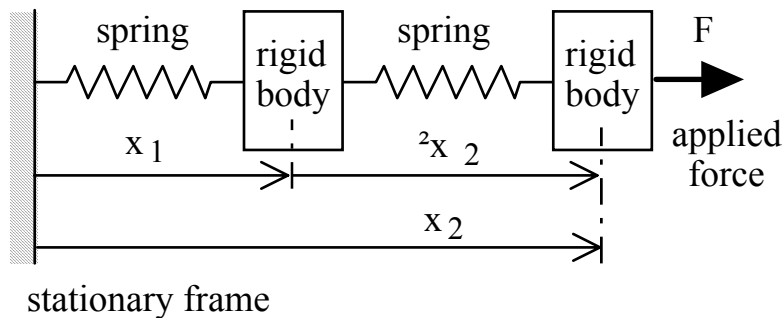


Figure 6.2a

A mechanical system schematic is shown in figure 6.2a. This schematic represents a model of the system and specifies the lumped-parameter energetic elements to be used. A corresponding bond graph is shown in figure 6.2b. It is evident that a single displacement or speed is common to the leftmost spring and inertia which are therefore connected by a one junction. Similarly, a single displacement or speed is common to the rightmost inertia and the source of the applied force which are therefore also connected by a one junction. The force in the remaining spring is applied to both inertias and a zero junction connects these three elements as shown. Note that a sign convention is chosen so that the displacement of the rightmost spring is determined by the difference of the displacements of the two inertias.

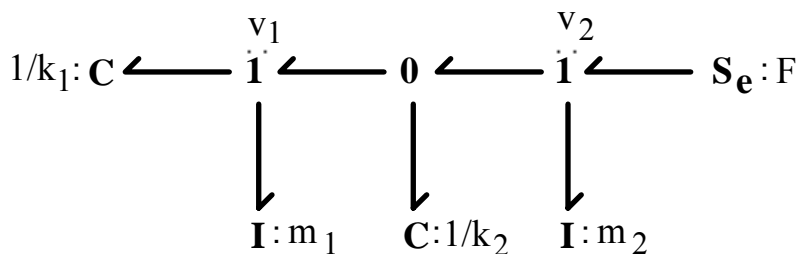


Figure 6.2b

Assume all elements have linear constitutive equations. Causality is assigned following the procedure described above with the result shown in figure 6.2c.

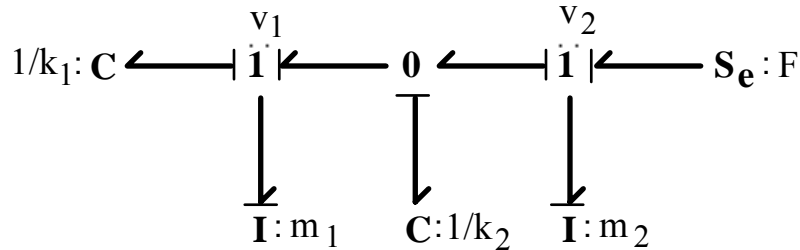


Figure 6.2c

From the causal graph, all energy storage elements are independent. Choose as state variables the speeds of the two inertias, v_1 and v_2 , and the corresponding displacements, x_1 and x_2 , shown in figure 6.2a. The first two state equations are:

$$\frac{d}{dt} x_1 = v_1 \quad (6.25)$$

$$\frac{d}{dt} x_2 = v_2 \quad (6.26)$$

Subscripting variables in an obvious way, from the constitutive equations of the inertias we may write:

$$\frac{d}{dt} v_1 = \frac{F_1}{m_1} \quad (6.27)$$

$$\frac{d}{dt} v_2 = \frac{F_2}{m_2} \quad (6.28)$$

Reading from the causal graph, the force on the leftmost inertia is determined by the two springs.

$$F_1 = F_{k2} - F_{k1} \quad (6.29)$$

The constitutive equations of the springs relate these forces to the corresponding displacements.

$$F_{k1} = k_1 x_1 \quad (6.30)$$

$$F_{k2} = k_2 \Delta x_2 \quad (6.31)$$

The displacement of the rightmost spring may be related to the positions of the inertias by reading directly from the causal graph.

$$\Delta x_2 = x_2 - x_1 \quad (6.32)$$

Substituting yields a third state equation.

$$\frac{d}{dt} v_1 = \frac{1}{m_1} [k_2(x_2 - x_1) - k_1 x_1] \quad (6.33)$$

Reading from the causal graph, the force on the rightmost inertia is determined by the associated spring and the applied force.

$$F_2 = F - F_{k_2} \quad (6.34)$$

Substituting the constitutive equation of the spring yields the fourth state equation.

$$\frac{d}{dt} v_2 = \frac{1}{m_2} [F - k_2(x_2 - x_1)] \quad (6.35)$$

The state equations may be written in standard vector/matrix form as follows.

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & 0 & 0 \\ k_2/m_2 & -k_2/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} F \quad (6.36)$$