

Examples: First-Order Systems

Energy storage elements provide the basis of the state equations we will derive to describe the dynamic processes occurring in a system. Of course, an energy storage element does not by itself define a dynamic process — it needs an input. That input will arise from the interaction with other system components as we will see in the following examples.

A simple fluid system

The sketch in figure 4.7 depicts an open-topped cylindrical container of water with a section of circular pipe connected at the bottom through which may leave. We wish to predict the time-course of the volume of fluid as the container empties.

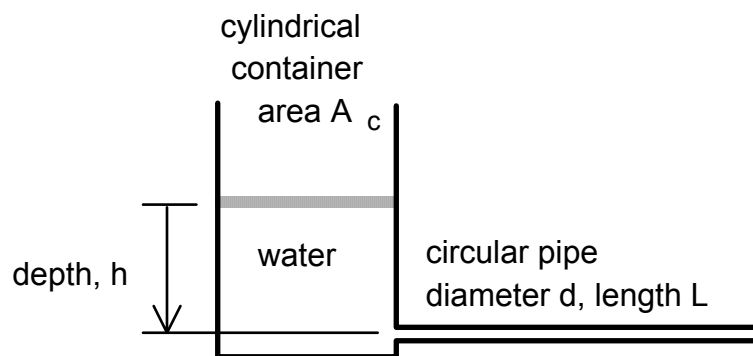


Figure 4.7: A simple fluid system.

At any cross section of interest, the instantaneous power transmitted is the product of the flow velocity and the force exerted on the cross section¹. The force, F , exerted on the cross section is the product of pressure², P_g , and area, A

$$F = P_g A \quad (4.24)$$

The volumetric flow rate, Q , is the product of flow velocity, v , and area.

$$Q = v A \quad (4.25)$$

Thus the power transmitted is the product of pressure and volumetric flow rate.

¹ Strictly speaking, we may only refer to a unique flow velocity, force, pressure, etc. if the cross-section has infinitesimal area. If the area is finite, each of the quantities represents an average over the cross section.

² For this example, it is assumed that we are dealing with *gage* pressures, i.e. that atmospheric pressure is taken to be zero, hence the subscript g .

$$P = F v = P_g Q \quad (4.26)$$

Within the cylindrical container, the pressure at any depth, h , (e.g. the depth of the entry to the pipe) is due to the weight, W , of the water above.

$$P_c = W/A_c = \rho g h A_c / A_c = \rho g h \quad (4.27)$$

where subscript c denotes the container, ρ is the density of water, g is gravitational acceleration and A_c is the area of the container. We may rewrite this relation in terms of the volume of water, V_c , above a given depth.

$$P_c = \frac{\rho g}{A_c} V_c \quad (4.28)$$

Consequently, we may describe the container as an ideal linear capacitor, characterized by a linear relation between pressure (effort) and volume (displacement). The capacitance, C , is determined by the geometric and material properties of the container, the fluid, etc.

$$C = \frac{A_c}{\rho g} \quad (4.29)$$

The energy stored in the capacitor is determined by the displacement variable, V_c , and the system parameters.

$$E_c = \frac{\rho g}{2 A_c} V_c^2 \quad (4.30)$$

The flow rate through the pipe is determined by the pressure difference between its ends. If we assume the pipe is horizontal with a constant circular cross-section and the flow is laminar, we may describe the pressure/flow-rate relation using the Hagen-Poiseuille law.

$$P_p = \frac{128 L \mu}{\pi d^4} Q_p \quad (4.31)$$

where P is the gage pressure at the end of the pipe next to the container, subscript p denotes the pipe, L is the length of the pipe, d is its diameter and μ is the absolute viscosity of water. This is the equation of an ideal linear resistor with resistance, R , determined by the geometric and material properties of the pipe, the fluid, etc.

$$R = \frac{128 L \mu}{\pi d^4} \quad (4.32)$$

The remaining idealized element in this system is the junction between the pipe and the container. If we assume the pressure at the entry to the pipe is the same as the pressure in the container at that depth, that common pressure defines a zero junction and a bond graph is as shown in figure 4.8.

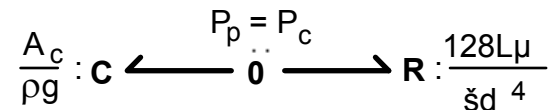


Figure 4.8: Bond graph of the simple fluid system.

Given this graph it is straightforward to determine state equations. We are interested in the rate of the change of the volume in the container which (from the definition of a generalized displacement) is due to the flow rate into it.

$$dV_c/dt = Q_c \quad (4.33)$$

That flow rate is determined from the continuity equation of the zero junction. Reading signs from the half arrows on the graph:

$$Q_c = - Q_p \quad (4.34)$$

The flow rate in the pipe is determined by inverting equation 4.30.

$$Q_p = P_p/R \quad (4.35)$$

From the zero junction definition, the pipe pressure is the same as the container pressure.

$$P_p = P_c \quad (4.36)$$

The container pressure is determined from equation 4.28. Successively substituting (4.28 into 4.35 into 4.34 into 4.32) yields a first-order linear state equation.

$$dV_c/dt = -V_c/RC \quad (4.37)$$

Note that this simple system has one energy-storage element and is characterized by a first-order state equation. The state variable, V_c , is directly related to the stored energy. This simple state equation may readily be integrated.

$$\int_{t_0}^t dV_c/V_c = \int_{t_0}^t -dt/RC \quad (4.38)$$

$$\ln\{V_c(t)/V_c(t_0)\} = (t_0 - t)/RC \quad (4.39)$$

$$V_c(t) = V_c(t_0)e^{(t_0 - t)/RC} \quad (4.40)$$

Note that to predict the behavior of this first-order system we require one initial condition which is related to the energy stored at time t_0 .

A simple electrical system

Figure 4.9 shows a diagram of a simple electrical circuit consisting of a capacitor connected to a resistor.



Electrical circuit diagram

Figure 4.9: A simple electrical system and a corresponding bond graph.

Assuming a common voltage drop across the resistor and capacitor, a corresponding bond graph is also shown, which is the same as the bond graph of the previous example. This is no accident: the two systems exhibit analogous energetic behavior.

Assuming an ideal linear capacitor, its charge, q , is proportional to the voltage difference, e , across it.

$$e = q/C \tag{4.41}$$

where the capacitance, C , is a physical property of the device. The rate of change of charge is the current, i_C , flowing into the capacitor.

$$dq/dt = i_C \tag{4.42}$$

Assuming an ideal linear electrical resistor, the current through it, i_R , is proportional to the voltage difference across it, e , as determined by Ohm's law

$$i_R = e/R \tag{4.43}$$

where the resistance, R , is a physical property of the device.

A zero junction describes the interaction between these elements. Its associated flow continuity equation requires that the current into the capacitor is the negative of the current into the resistor.

$$i_C = -i_R \tag{4.44}$$

One reasonable choice of state variable is the charge on the capacitor. A first-order state equation is obtained by substitution (4.41 into 4.43 into 4.44 into 4.42).

$$dq/dt = -e/RC \tag{4.45}$$

Assuming the obvious analogy between like quantities (voltage drop and pressure difference are both effort variables, charge and volume are both displacement variables, etc.) equations 4.45 and 4.37 are clearly similar; the time response for discharging the capacitor is of the same form as the time response for emptying the fluid container and will be described by an equation similar to equation 4.40.

The analogous dynamic behavior of these two systems is represented by their similar bond graphs. In this regard, bond graphs provide a unified notation for depicting different physical systems.

A nonlinear fluid system

In this example consider a system similar to that of figure 4.7 but instead of a container with a constant cross-sectional area, assume the radius, r , of the circular cross-section varies with height as follows.

$$r = ah^n \quad n \geq 0 \quad (4.46)$$

where n is a positive constant and a is a scaling factor. The volume varies with depth as follows.

$$V_c = \int_0^h \pi a^2 h^{2n} dh = \frac{\pi a^2 h^{2n+1}}{2n+1} \quad (4.47)$$

Pressure varies with depth as before (equation 4.27) and the relation between pressure and volume may be obtained by substitution.

$$P_c = \rho g \left[\frac{2n+1}{\pi a^2} V_c \right]^{\frac{1}{2n+1}} \quad (4.48)$$

Whereas in the previous example the container was described as an ideal linear capacitor, in this example it may be described as an ideal capacitor.

The same bond graph (figure 4.8 or 4.9) may be used to represent this system too, the only difference being that the capacitor is characterized by a nonlinear constitutive equation. Indeed, the special case $n = 0$ corresponds to a container with a constant cross-sectional area; substituting $n = 0$, equation 4.48 reduces to equation 4.28. A state equation may be obtained following the same procedure as above but using equation 4.48 in place of equation 4.28.

$$\frac{dV_c}{dt} = -\frac{\rho g}{R} \left[\frac{2n+1}{\pi a^2} V_c \right]^{\frac{1}{2n+1}} \quad (4.49)$$

As before, this system contains a single (nonlinear) energy-storage element and is characterized by a first-order (nonlinear) state equation. Once again, it is again straightforward to integrate. The case $n = 0$ has been considered above. For $n > 0$ the result is as follows.

$$\int_{t_0}^t V_c^{-\frac{1}{2n+1}} dV_c = \int_{t_0}^t -\frac{\rho g}{R} \left[\frac{2n+1}{\pi a^2} \right]^{\frac{1}{2n+1}} dt \quad (4.50)$$

$$V_c(t)^{\frac{2n}{2n+1}} = V_c(t_0)^{\frac{2n}{2n+1}} + \frac{2n}{2n+1} \frac{\rho g}{R} \left[\frac{2n+1}{\pi a^2} \right]^{\frac{1}{2n+1}} (t_0 - t) \quad (4.51)$$

$$V_c(t) = \left\{ V_c(t_0)^{\frac{2n}{2n+1}} + \frac{2n}{2n+1} \frac{\rho g}{R} \left[\frac{2n+1}{\pi a^2} \right]^{\frac{1}{2n+1}} (t_0 - t) \right\}^{\frac{2n+1}{2n}} \quad (4.52)$$

Though rather clumsy-looking, this equation, valid for $V_c \geq 0$, yields the simple behavior shown in figure 4.10.

Comparing these examples, some points should be noted:

A bond graph such as that of figure 4.8 or 4.9 is an abstract representation of a *family* of systems. Until the constitutive equations of the elements have been specified, state equations cannot be determined. However, all members of the family of systems represented by the bond graph share certain qualitative features. In the above examples, first-order state equations are sufficient to describe energetic transactions within the system; that is a consequence of the single energy-storage element. The behavior of the systems are qualitatively similar; all exhibit a non-oscillating decay to the equilibrium state $V_c = 0$ or $q = 0$.

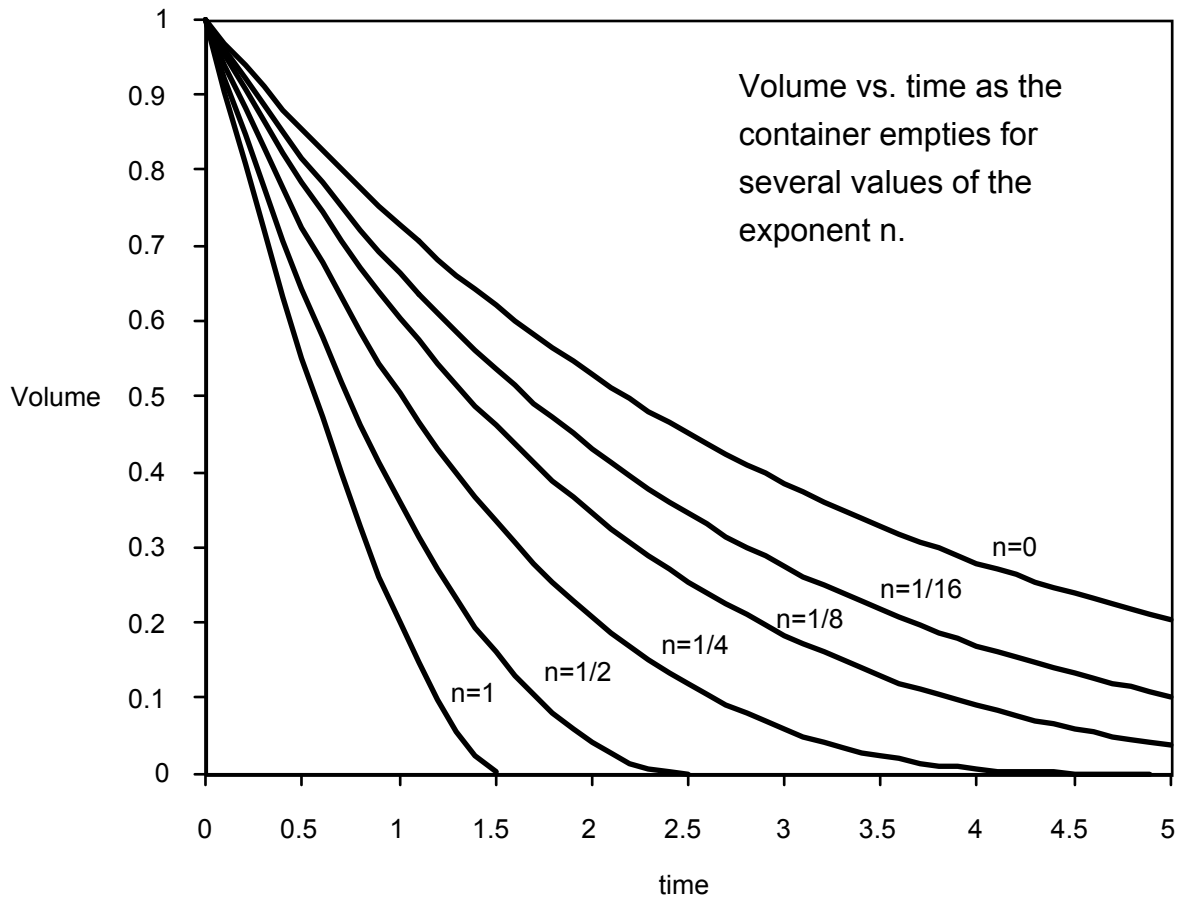


Figure 4.10: Time course of emptying for different container shapes. Fixed parameters have been assigned arbitrary values.

It is a common misconception to regard linear and nonlinear systems as radically different. A more enlightened perspective is to consider linear systems as special cases with certain interesting properties. In the examples above all of the nonlinear systems make the reasonable prediction that a finite time is required to empty the container. From equation 4.51 the time to empty is given by

$$t_{\text{empty}} = t_0 + \frac{V_c(t_0)^{\frac{2n}{2n+1}}}{\frac{2n}{2n+1} \frac{\rho g}{R} \left[\frac{2n+1}{\pi a^2} \right]^{\frac{1}{2n+1}}} \quad n \neq 0 \quad (4.53)$$

In contrast, it can be seen from equation 4.39 that the time for the linear system to empty is infinite. But as the exponent n becomes small the time to empty becomes large and the same result could be obtained by taking n to zero in the limit in equation 4.53.

$$\lim_{n \rightarrow 0} t_{empty} = \infty \quad (4.54)$$

Another interesting property of the linear case is that the responses starting from different initial conditions are all of the same form; that is, they are identical when multiplied by a scaling factor inversely proportional to the initial condition. This is clearly not the case for the nonlinear systems: The value of a response at each point in time can be regarded as the initial condition for that response for all future time; yet for sufficiently large times the nonlinear responses are identically zero and therefore cannot be scaled to match any non-zero response.