

MIT OpenCourseWare  
<http://ocw.mit.edu>

2.161 Signal Processing: Continuous and Discrete  
Fall 2008

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 Signal Processing - Continuous and Discrete  
Fall Term 2008

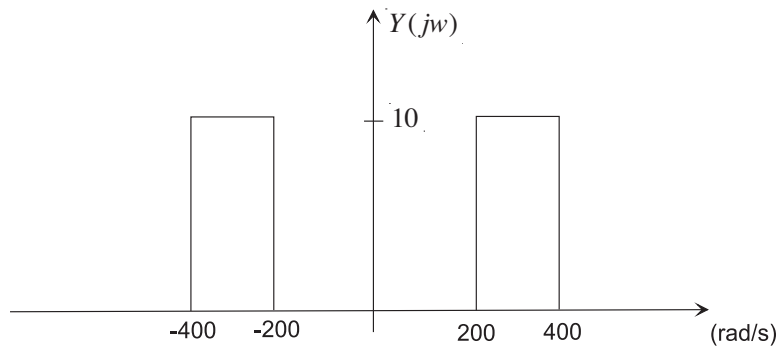
**Solution of Problem Set 2**

**Assigned:** Sept. 18, 2008

**Due:** Sept. 25, 2008

**Problem 1:**

$Y(j\Omega) = H(j\Omega)F(j\Omega)$  and it is shown in the below figure



$$\begin{aligned}
 y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\Omega) e^{j\Omega t} dt \\
 &= \frac{10}{2\pi} \left( \int_{200}^{400} e^{j\Omega t} dt + \int_{-400}^{-200} e^{j\Omega t} dt \right) \\
 &= \frac{10}{2\pi jt} (e^{400jt} - e^{200jt} - e^{-400jt} + e^{-200jt}) \\
 &= \frac{10}{\pi t} (\sin(400t) - \sin(200t))
 \end{aligned}$$

It is not a casual filter (since  $y(t) = h(t)/5$  and  $h(t) \neq 0$  for some  $t < 0$ ).

**Problem 2:**

We define  $H^*(j\Omega) = 1 - H(j\Omega)$ . Then  $H^*(j\Omega)$  is a low pas filter, matching Prob. 5 in PS 1, which we have already found it's impulse response:

$$\begin{aligned}
 H(j\Omega) &= 1 - H^*(j\Omega) \\
 \mathcal{F}^{-1}(H(j\Omega)) &= \mathcal{F}^{-1}(1) - \mathcal{F}^{-1}(H^*(j\Omega))
 \end{aligned}$$

$$\begin{aligned}
 h(t) &= \delta(t) - h^*(t) \\
 h(t) &= \delta(t) - \frac{\sin(\Omega_c t)}{\pi t}
 \end{aligned}$$

**Problem 3:**

$$x(t) = \sum_{n=0}^{\infty} A_n \sin(n\Omega_0 t + \phi_n) = \sum_{n=0}^{\infty} A_n (\sin(n\Omega_0 t) \cos \phi_n + \cos(n\Omega_0 t) \sin \phi_n)$$

$$X(j\Omega) = \sum_{n=0}^{\infty} A_n (\mathcal{F}(\sin(n\Omega_0 t)) \cos \phi_n + \mathcal{F}(\cos(n\Omega_0 t)) \sin \phi_n)$$

$$X(j\Omega) = \sum_{n=0}^{\infty} A_n (\cos \phi_n (-j\pi (\delta(\Omega - n\Omega_0) - \delta(\Omega + n\Omega_0))) + \sin \phi_n (\pi (\delta(\Omega - n\Omega_0) + \delta(\Omega + n\Omega_0))))$$

$$X(j\Omega) = -j\pi \sum_{n=0}^{\infty} A_n ((\cos \phi_n + j \sin \phi_n) (\delta(\Omega - n\Omega_0)) + (-\cos \phi_n + j \sin \phi_n) (\delta(\Omega + n\Omega_0)))$$

$$X(j\Omega) = -j\pi \sum_{n=0}^{\infty} A_n (e^{j\phi_n} \delta(\Omega - n\Omega_0) - e^{-j\phi_n} \delta(\Omega + n\Omega_0))$$

**Problem 4:**

- (a) The ideal multiplicative filtering operation is a low pass filtering with the pass-band  $\Omega_c = N\Omega_0$ :

$$H_n = \begin{cases} 1 & |n| \leq N, \quad |\Omega| \leq \Omega_c \\ 0 & |n| > N, \quad |\Omega| > \Omega_c \end{cases}$$
$$\hat{X}_n = X_n H_n$$

- (b) It's a convolution in this specific form:

$$\hat{x}(t) = x(t) \otimes h(t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau) h(t - \tau) d\tau$$

We can prove that why convolution is in this specific integral form for our Periodic Exponential Fourier Transform:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} (X_n H_n e^{jn\Omega_0 t}) = \sum_{n=-\infty}^{\infty} \left( X_n \left( \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} h(\tau) e^{-jn\Omega_0 \tau} d\tau \right) e^{jn\Omega_0 t} \right)$$

$$\hat{x}(t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left( h(\tau) \sum_{n=-\infty}^{\infty} (X_n e^{-jn\Omega_0 \tau} e^{jn\Omega_0 t}) \right) d\tau = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left( h(\tau) \sum_{n=-\infty}^{\infty} (X_n e^{jn\Omega_0 (t-\tau)}) \right) d\tau$$

$$\hat{x}(t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} h(\tau) x(t - \tau) d\tau$$

We can also find the specific form of our filter:

$$h(t) = \sum_{n=-\infty}^{\infty} H_n e^{jn\Omega_0 t} = \sum_{n=-N}^N e^{jn\Omega_0 t} = 1 + 2 \sum_{n=1}^N \cos(n\Omega_0 t) = -1 + 2 \sum_{n=0}^N \cos(n\Omega_0 t)$$

Here we use below trigonometric relation to simplify our summation, where  $\varphi = 0$  and  $\alpha = \Omega_0 t$ :

$$\cos \varphi + \cos (\varphi + \alpha) + \cos (\varphi + 2\alpha) + \cdots + \cos (\varphi + n\alpha) = \frac{\sin \left( \frac{(n+1)\alpha}{2} \right) \cdot \cos \left( \varphi + \frac{n\alpha}{2} \right)}{\sin \frac{\alpha}{2}}$$

$$h(t) = -1 + 2 \sum_{n=0}^N \cos (n\Omega_0 t) = -1 + 2 \frac{\sin \left( \frac{(N+1)\Omega_0 t}{2} \right) \cdot \cos \left( \frac{N\Omega_0 t}{2} \right)}{\sin \frac{\Omega_0 t}{2}}$$

Then we use below trigonometric relation to simplify our impulse response:

$$2 \sin \theta \cos \varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$$

$$h(t) = -1 + 2 \frac{\sin \left( \frac{(N+1)\Omega_0 t}{2} \right) \cdot \cos \left( \frac{N\Omega_0 t}{2} \right)}{\sin \frac{\Omega_0 t}{2}} = -1 + \frac{\sin \left( (N + \frac{1}{2})\Omega_0 t \right) + \sin \left( \frac{\Omega_0 t}{2} \right)}{\sin \frac{\Omega_0 t}{2}}$$

$$h(t) = \frac{\sin \left( (N + \frac{1}{2})\Omega_0 t \right)}{\sin \frac{\Omega_0 t}{2}}$$

Here we further analyze this  $h(t)$  function, so we can use its properties for the next part of the problem:

Note that for small  $t$  values,  $h(t)$  can be approximated to:

$$t \rightarrow 0 \Rightarrow h(t) \rightarrow \frac{\sin \left( (N + \frac{1}{2})\Omega_0 t \right)}{\frac{\Omega_0 t}{2}}$$

The value of  $h(t)|_{t \rightarrow 0}$ , for large  $N$  values, is very close to  $h^* = \frac{\sin(N\Omega_0 t)}{\pi t}$  which can be obtained for a *Continuous* Fourier Transform of a Low Pass Filter with  $\Omega_c = N\Omega_0$ . Hence, in the vicinity of  $t = 0$ ,  $h(t)$  acts like a sinc function (with the maximum value of  $2N + 1$ ), but as soon as it gets close to its boundaries ( $|t| \leq \frac{T}{2} \Rightarrow |\Omega_0 t| \leq \pi$ ), it oscillates quickly with  $\Omega = (N + 0.5)\Omega_0$  around  $-1$  and  $+1$ .

Note that for the *Continuous* Fourier Transform, we expect a  $h^*(t)$ ; where it is to be evaluated between  $-\infty$  to  $+\infty$  and contained with an envelope in the form of  $\frac{1}{t}$ . On the other hand, for our case of *Periodic* Fourier Transform,  $h(t)$  it is to be evaluated between  $-\frac{T}{2}$  to  $+\frac{T}{2}$  and is also periodic.

Note that for any  $N$  value, if  $x(t) \equiv 1$  then  $\hat{x}(t) \equiv 1$  which means that following relation holds for any  $N$ :

$$1 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} h(\tau) d\tau = \frac{2}{T} \int_0^{\frac{T}{2}} h(\tau) d\tau$$

Furthermore, for very large  $N$  values, the  $h(t)$  function become very narrow, and hence above integral can be approximated to the integration around any finite, non-zero interval around  $t = 0$ :

$$\text{(Eq. 1)} \quad N \rightarrow \infty \Rightarrow \forall t^* \text{ s.t. } 0 < t^* \leq \frac{T}{2} : \frac{1}{T} \int_0^{t^*} h(\tau) d\tau \rightarrow \frac{1}{T} \int_0^{\frac{T}{2}} h(\tau) d\tau$$

This properties can be used to prove that  $\hat{x}(t)$  values converge to  $x(t)$  values at any continuous point and to the mean of right and left limits at any stepwise discontinuous point of  $x(t)$ . Furthermore, the structure of  $h(t)$  shows that the ripple will vanish at any point in which  $x(t)$  is continuous. Besides, as we will see in the next part, only a finite and determined value of ripple will be allowed to remain and it will be pushed to the edge of discontinuities.

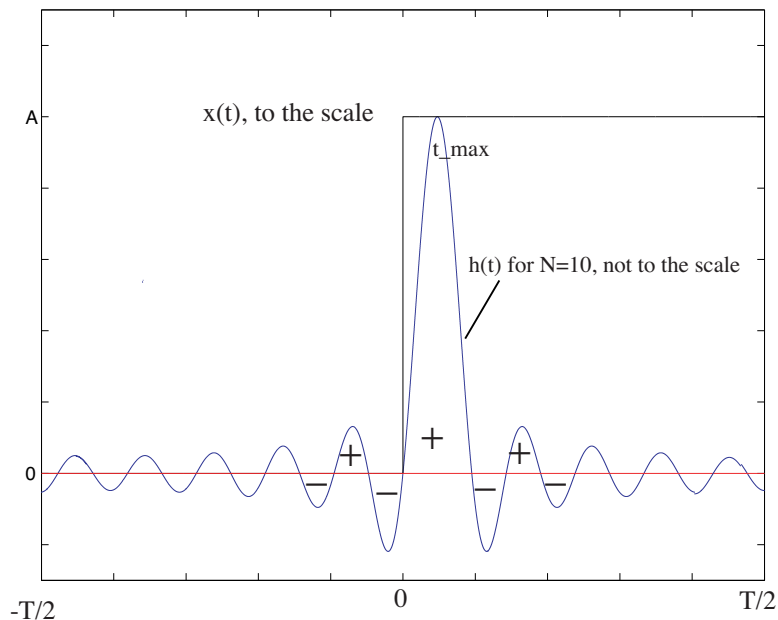
- (c) The output signal is a convolution of the original signal with the  $h(t)$ . Hence, it is very clear that ripples are due to deviation of  $h(t)$  from ideal case of  $\delta(t)$ . Consequently, as soon as  $N$  goes up,  $h(t)$  becomes more like a  $\delta(t)$  function, acts more locally, and the output signal ripples with an increases frequency.

For very large  $N$  values, the ripple percentage is a fixed amount, although its location is dependent on  $N$  value. The amount of ripple is fixed, because although different  $h(t)$  functions have different curves, but always the maximum ripple corresponds to the point where the edge of the central lobe matches with the discontinuity. We will prove this rigorously, but in fact since the ratio of the area of the central lobe, to the rest of the lobes remains constant, the ripple percentage remains constant as well.

Note that due to Eq. 1, only the behavior of function around discontinuity matters (as long as discontinuities have a finite non-zero distance from each other). Hence, without the loss of generality, we extend discontinuity limits to the the full extent of the function and consider our function in the below form where  $A$  value corresponds to the discontinuity jump:

$$x(t) = \begin{cases} A & 0 \leq t < \frac{T}{2} \\ 0 & -\frac{T}{2} \leq t < 0 \end{cases}$$

Below figure shows this conditions:



By moving  $h(t)$  on the  $x(t)$ , we can realize that, at each point, the  $\hat{x}(t)$  is an average of neighborhood points. Particularly, the sign of lobes determine that whether those

point have a positive or negative contribution. Since, major contribution comes from the central lobes, we can guess that the maximum ripple occurs when the edge of central lobe match with  $t = 0$ . In that case, the next negative lobe cannot decrease the  $\hat{x}(t)$  value and we have a maximum ripple. The next maxima and minima of ripples also correspond to other lobe edges touching  $t = 0$ .

Now we prove this rigorously. For our specific case of  $x(t)$ , and our even  $h(t)$  function, we can simplify the convolution integral:

$$\hat{x}(t) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} h(t - \tau)x(\tau)d\tau = \frac{A}{T} \int_0^{\frac{T}{2}} h(t - \tau)d\tau = \frac{A}{T} \int_{-t}^{\frac{T}{2}-t} h(\tau)d\tau$$

$$\frac{d\hat{x}(t)}{dt} = -h\left(\frac{T}{2} - t\right) + h(t)$$

We are interested in  $0 < t \ll \frac{T}{2}$ , and hence  $\frac{d\hat{x}(t)}{dt} \approx h(t)$ . Consequently, maximum/minimum values correspond to  $(N + \frac{1}{2})\Omega_0 t_{max/min} = k\pi$  such that  $0 < k \ll N$ . The highest maximum corresponds to the  $t_{max} = \frac{\pi}{(N + \frac{1}{2})\Omega_0} = \frac{T}{2(N + \frac{1}{2})}$ . Now we compute the  $\hat{x}(t_{max})$  value by breaking the original integral to two parts:

$$\hat{x}(t_{max}) = \frac{A}{T} \int_{-t_{max}}^{\frac{T}{2}-t_{max}} h(\tau)d\tau = A\left(\frac{1}{T} \int_{-t_{max}}^0 h(\tau)d\tau + \frac{1}{T} \int_0^{\frac{T}{2}-t_{max}} h(\tau)d\tau\right)$$

Now note that for large N values from Eq. 1 we can conclude that:

$$\frac{1}{T} \int_0^{\frac{T}{2}-t_{max}} h(\tau)d\tau = \frac{1}{T} \int_0^{\frac{T}{2}} h(\tau)d\tau = \frac{1}{2}$$

Also for the other part of integral,  $\tau$  is very close to zero and we can simplify  $h(t)$  with a sinc form and also change integration variable by  $\theta = (N + \frac{1}{2})\Omega_0 t$ :

$$\frac{1}{T} \int_{-t_{max}}^0 h(\tau)d\tau = \frac{1}{T} \int_{-t_{max}}^0 \frac{\sin((N + \frac{1}{2})\Omega_0\tau)}{\frac{\Omega_0\tau}{2}}d\tau = \frac{1}{T} \int_{-\pi}^0 \frac{\sin\theta}{\theta} \frac{2}{\Omega_0}d\theta = \frac{1}{\pi} \int_0^\pi \frac{\sin\theta}{\theta}d\theta$$

A short survey of literature or a Taylor expansion of  $\frac{\sin\theta}{\theta}$  ends to:

$$\frac{1}{\pi} \int_0^\pi \frac{\sin\theta}{\theta}d\theta = \frac{1}{2} + 0.089490\dots$$

Hence the maximum ripple corresponds to:

$$\hat{x}(t_{max}) = A\left(\frac{1}{2} + \frac{1}{2} + 0.089490\dots\right) = A(1 + 0.089490\dots)$$

This corresponds to about 9% overshoot and this overshoot is only dependent on the local discontinuity jump and independent of  $N$  or specific form of  $x(t)$ . Other proofs, for less general cases, could also be found in the Wikis.

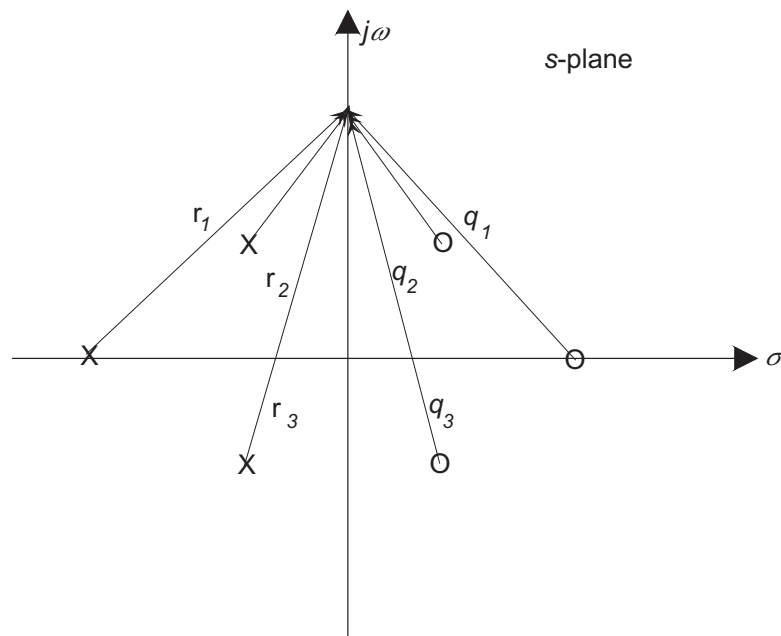
### Problem 5:

(a)

$$H(s) = K \frac{s - 3}{s + 3}$$

$$|H(j\Omega)| = |K| \left| \frac{j\Omega - 3}{j\Omega + 3} \right| = |K|$$

(b) Consider this pole-zero plot:



Let the vectors from the poles and zeros to an arbitrary test frequency  $\Omega$  be  $r_i$  and  $q_i$ :

$$|H(j\Omega)| = |K| \frac{|q_1| |q_2| |q_3|}{|r_1| |r_1| |r_1|} = |K|$$

since  $|q_i| = |r_i|$  for all  $i$  regardless of the system order. Therefore, this given pole-zero configuration, as well as all who satisfy problem conditions, are all-pass filters.

(c) **MATLAB Command – line :**

```
>> H_s=tf([1 -3],[1 3])
```

Transfer function:

```
s - 3
```

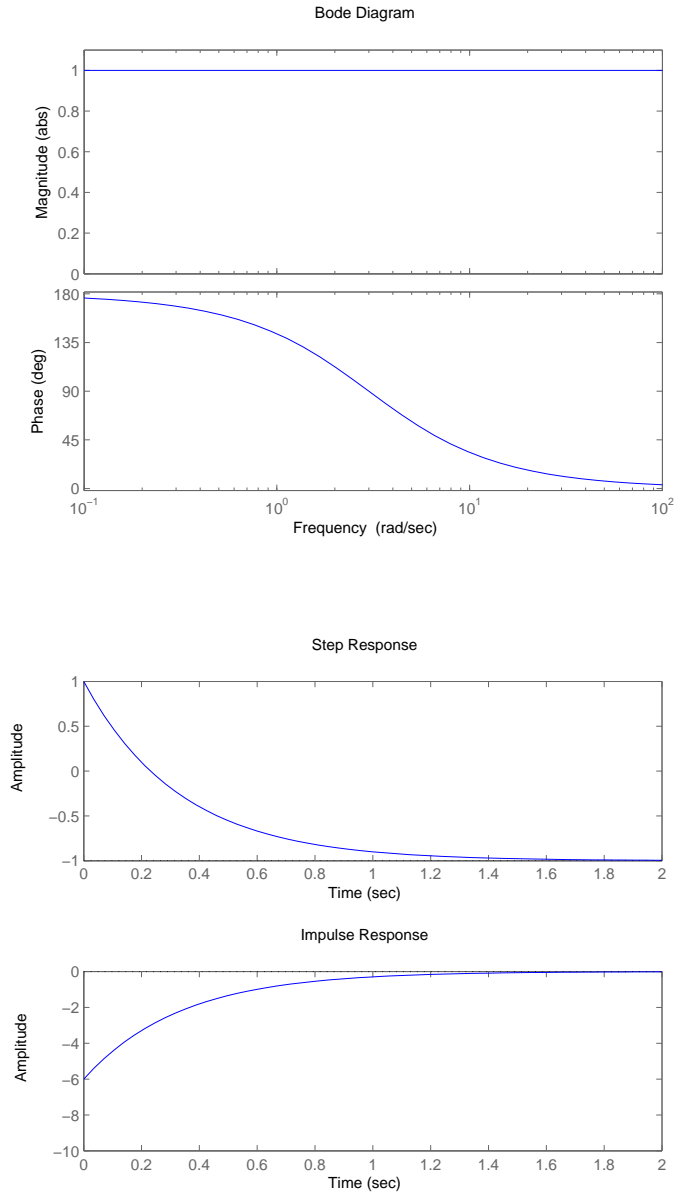
```
-----
```

```
s + 3
```

```
>> bode(H_s);
```

```
>> subplot(2,1,1)
```

```
>> step(H_s);  
>> subplot(2,1,2)  
>> impulse(H_s);
```



(d) These filters are useful for manipulating the phase of the spectral components in a signal, without altering its magnitude.