2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

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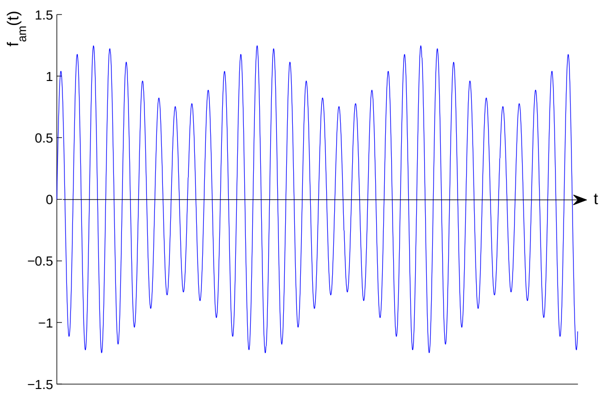
Problem Set 4

Assigned: Oct. 2, 2008 **Due:** Oct. 9, 2008

Problem 1: An AM (amplitude-modulated) radio signal $f_{AM}(t)$ is described by

$$f_{AM}(t) = (1 + af_{audio}(t)) \sin(\Omega_c t)$$

where $f_{audio}(t)$ is the audio signal, $\sin(\Omega_c t)$ is known as radio-frequency *carrier* signal ($f_c =$ 500 – 1600 kHz - the AM band), and a is a positive constant that determines the *modulation depth*. (Note that we require $|af_{audio}(t)| < 1$ otherwise we have over-modulation.) the following figure shows an AM signal with an "audio" waveform that is a simple low frequency sinusoid. You can see how the audio signal "modulates" the amplitude of the rf signal.



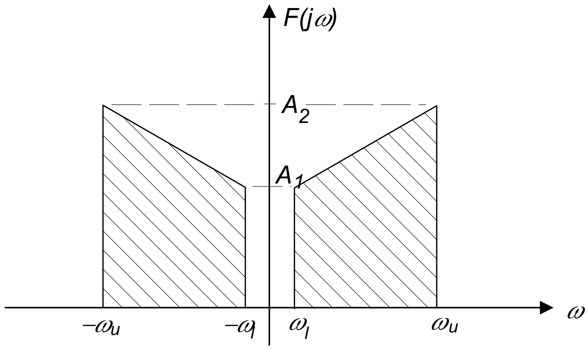
(a) Sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when $f_{audio}(t) = 0$.

(b) Let $a = 0.5$, and sketch the magnitude of the Fourier transform of $f_{AM}(t)$ when

$$f_{audio}(t) = 0.5\cos(2\pi \cdot 1000t) + 0.25\cos(2\pi \cdot 2000t)$$

(**Hint:** There is no need to actually compute the FT. Consider expanding $f_{AM}(t)$, or simply use properties of the FT.)

(c) Use your result from (b) to generalize, and sketch the magnitude spectrum of $f_{AM}(t)$ when $f_{audio}(t)$ has a spectrum (again let $a = 0.5$):



(d) If f_{audio} has a bandwidth $B = \Omega_u - \Omega_l$, what is the bandwidth of a band-pass filter that would be necessary to select the signal $f_{AM}(t)$ out of all the other AM radio stations?

Problem 2: We generally ignore in the phase response in filter design. Although you might wish for a "zero-phase" filter, you can see from the class handout on causality that a filter with a purely real frequency response is acausal, and as such cannot be implemented in a physical system. The following are a pair of tricks that may be used to do "off-line" zero-phase filtering of recorded data. (Note: these methods are used frequently in digital signal processing - it is difficult to do this in continuous time.)

Assume that you have a filter $H(j\Omega)$ with arbitrary phase response $\angle H(j\Omega)$, and that your input signal is $f(t)$ is recorded on a tape-recorder that can be played forwards or backwards.

Method (1)

1. Pass $f(t)$ through the filter and record the output $g(t)$ on another tape recorder.
2. Play $g(t)$ backwards through the filter (that is the filter input is $g(-t)$) and record the output $x(t)$.
3. The filtered output is found by playing the $x(t)$ backwards, that is $y(t) = x(-t)$.

Method (2)

1. Pass $f(t)$ through the filter and record the output $g(t)$.
2. Reverse $f(t)$ so as to pass $f(-t)$ through the filter and generate $x(t)$.
3. Reverse $x(t)$ and sum with $g(t)$ to form the output $y(t) = g(t) + x(-t)$.

Show that both methods generate an overall filter that has no phase shift, and find the overall magnitude response $|H_{eq}(j\Omega)|$ in each case. Hint: $\mathcal{F}\{f(-t)\} = \bar{F}(j\Omega)$.

Problem 3: Problem 5 in Problem Set 2 examined an *all-pass* filter with a transfer function

$$H(s) = \frac{s-a}{s+a} \qquad a > 0$$

and you showed that this filter had a frequency response in which $|H(j\Omega)| = 1$ at all frequencies.

Design an op-amp based first-order all-pass filter of this form that will have a phase shift of $-90°$ at a frequency of 50 Hz. (Consider a modified form of the 3 op-amp circuit described in the handout - noting that you only need a first-order system.) Find "appropriate" values for all resistors and capacitors.

Problem 4: Consider the second-order bandpass filter with transfer function

$$H_{bp}(s) = \frac{a_1 s}{s^2 + a_1 s + a_0}.$$

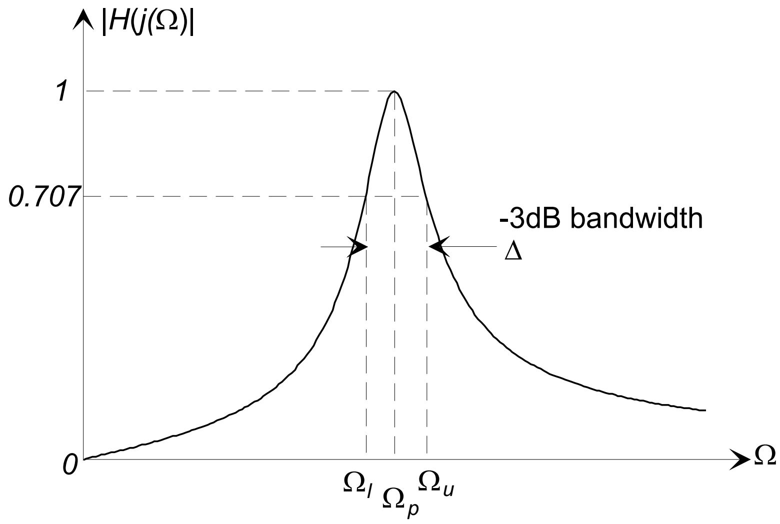
Many books on signal processing express this transfer function in terms of two parameters Ω_p and Q,

$$H_{bp}(s) = \frac{\frac{\Omega_p}{Q} s}{s^2 + \frac{\Omega_p}{Q} s + \Omega_p^2}$$

where Ω_p is the (approximate) peak frequency (center of the passband), and Q is known as the "quality" factor.

> **Aside:** If you compare this to the classic second-order system description used in system dynamics and control, that is
>
> $$H_{bp}(s) = \frac{2\zeta\Omega_n s}{s^2 + 2\zeta\Omega_n s + \Omega_n^2}$$
>
> where Ω_n is the undamped natural frequency, and ζ is the damping ratio, you can see that
>
> $$Q = \frac{1}{2\zeta}.$$

In this problem we examine the relationship between Q and the -3dB bandwidth of the second-order filter. Consider the magnitude plot below:



Let Ω_u and Ω_l be the upper and lower -3db (0.707) response frequencies as shown, and let $\Delta = \Omega_u - \Omega_l$ be the -3dB bandwidth.

(a) Show

$$\Omega_u = \Omega_p \sqrt{1 + \frac{1}{2Q^2} + \frac{1}{Q}\sqrt{1 + \left(\frac{1}{2Q}\right)^2}}$$
$$\Omega_l = \Omega_p \sqrt{1 + \frac{1}{2Q^2} - \frac{1}{Q}\sqrt{1 + \left(\frac{1}{2Q}\right)^2}}$$

(b) Use these results to show that the -3dB bandwidth of the second-order filter is

$$\Delta = \frac{\Omega_p}{Q}$$

Hint: Write $\Delta = \sqrt{a+b} - \sqrt{a-b}$.

(c) Determine the transfer function of a second-order bandpass filter with a center frequency of 100 Hz., and a -3dB bandwidth of 10 Hz.

Problem 5: A sampling system takes samples at regular intervals ΔT. Assume we have a sinusoid

$$y(t) = \sin \Omega t,$$

so that the samples are $y(n\Delta T) = \sin n\Omega\Delta T$. We know that if the frequency Ω is greater than the Nyquist frequency $\Omega_N = \pi/\Delta T$, the sample set is *aliased*.

Now consider two sinusoids, one $y(t) = \sin n\Omega_0 \Delta T$ with a frequency Ω_0 that is below the Nyquist frequency, and another with frequency Ω_1 **above** the Nyquist frequency.

(a) Assume $y_1(t) = \sin \Omega_1 t$ where

$$\Omega_1 = 2k\Omega_N - \Omega_0, \qquad k = 1 \ldots \infty \text{ is any positive integer.}$$

Show that the sample sets are related by $y_1(n\Delta T) = -y(n\Delta T) = -\sin(n\Omega_0\Delta T)$,

(b) Repeat part (a) with

$$\Omega_1 = 2k\Omega_N + \Omega_0, \qquad k = 1 \ldots \infty \text{ is any positive integer.}$$

and show that in this case the sample sets are identical, that is $y_2(n\Delta T) = y(n\Delta T) = \sin(n\Omega_0\Delta T)$.

(c) Use the results of (a) and (b) to graphically demonstrate the concept of "frequency folding" of aliased sinusoids.

(d) A periodic waveform is written as a Fourier Series

$$y(t) = 5\sin(2\pi(25)t) + 2sin(2\pi(75)t) + 3\sin(2\pi(125)t).$$

If the waveform is sampled at 100 samples/sec, determine the frequencies and amplitudes of the spectral components in the sampled waveform. (Hint: The results of parts (a) and (b) should help.)