2.161 Signal Processing: Continuous and Discrete
Fall 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

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Fall Term 2008

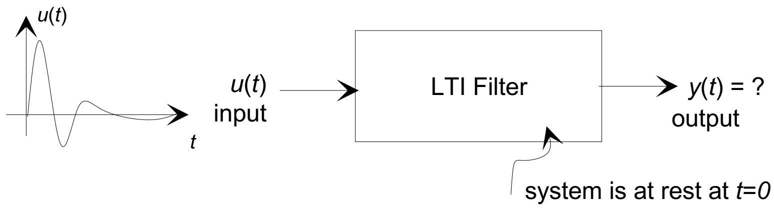
Lecture 2[1]

Reading:

- Class handout: Convolution
- Class handout: Sinusoidal Frequency Response

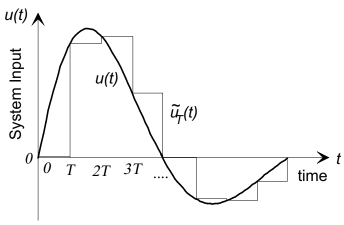
1 Continuous LTI System Time-Domain Response

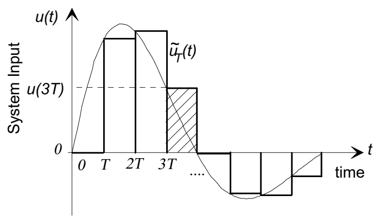
A continuous linear filter is a LTI dynamical system (described by an ODE with constant coefficients). We are interested in the input-output relationships and seek a method of determining the response $y(t)$ to a given input $u(t)$.



The relationship is developed as follows (see the handout for a detailed explanation)

- The input $u(t)$ is approximated as a zero-order (staircase) waveform $\tilde{u}_T(t)$ with intervals T.





$$\tilde{u}_T(t) = u(nT) \qquad \text{for } nT \le t < (n+1)T.$$

- The approximation $\tilde{u}_T(t)$ is written as a superposition of non-overlapping pulses

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} p_n(t)$$

where

$$p_n(t) = \begin{cases} u(nT) & nT \le t < (n+1)T \\ 0 & \text{otherwise} \end{cases}$$

For example, $p_3(t)$ is shown cross-hatched in the figure above.

- Each component pulse $p_n(t)$ is written in terms of a delayed *unit pulse* $\delta_T(t)$, of width T and amplitude $1/T$ that is:

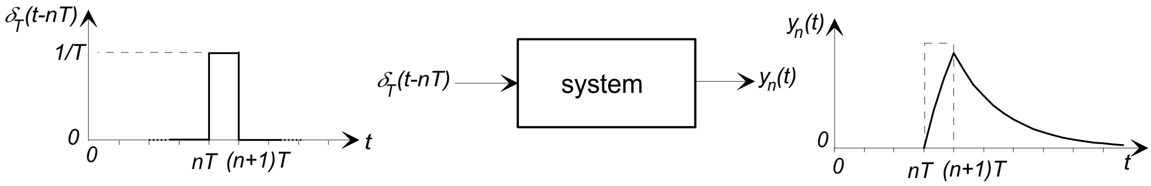
$$p_n(t) = u(nT)\delta_T(t - nT)T$$

so that

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)\delta_T(t - nT)T.$$

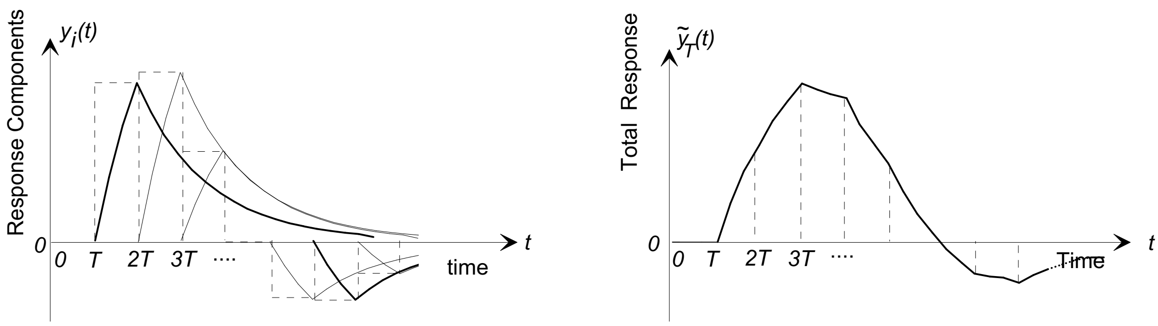
- Assume that the system response to an input $\delta_T(t)$ is a known function, and is designated $h_T(t)$ as shown below. If the system is linear and time-invariant, the response to a delayed unit pulse, occurring at time nT is simply a delayed version of the pulse response:

$$y_n(t) = h_T(t - nT)$$



- The principle of superposition allows the total system response to $\tilde{u}_T(t)$ to be written as the sum of the responses to all of the component weighted pulses:

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)h_T(t - nT)T$$



For causal systems the pulse response $h_T(t)$ is zero for time $t < 0$, and future components of the input do not contribute to the sum, so that the upper limit of the summation may be rewritten:

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{N} u(nT)h_T(t - nT)T \qquad \text{for } NT \le t < (N+1)T.$$

- We now let the pulse width T become very small, and write $nT = \tau$, $T = d\tau$, and note that $\lim_{T\to 0} \delta_T(t) = \delta(t)$. As $T \to 0$ the summation becomes an integral and

$$\begin{aligned} y(t) &= \lim_{T\to 0} \sum_{n=-\infty}^{N} u(nT) h_T(t - nT)T \\ &= \int_{-\infty}^{t} u(\tau)h(t-\tau)d\tau \end{aligned} \tag{1}$$

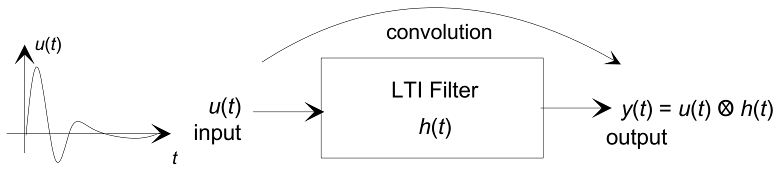
where $h(t)$ is defined to be the system *impulse response*,

$$h(t) = \lim_{T\to 0} h_T(t).$$

Equation (**??**) is an important integral in the study of linear systems and is known as the *convolution* or *superposition* integral. It states that the system is entirely *characterized* by its response to an impulse function $\delta(t)$, in the sense that the forced response to any arbitrary input $u(t)$ may be computed from knowledge of the impulse response alone. The convolution operation is often written using the symbol $\otimes$:

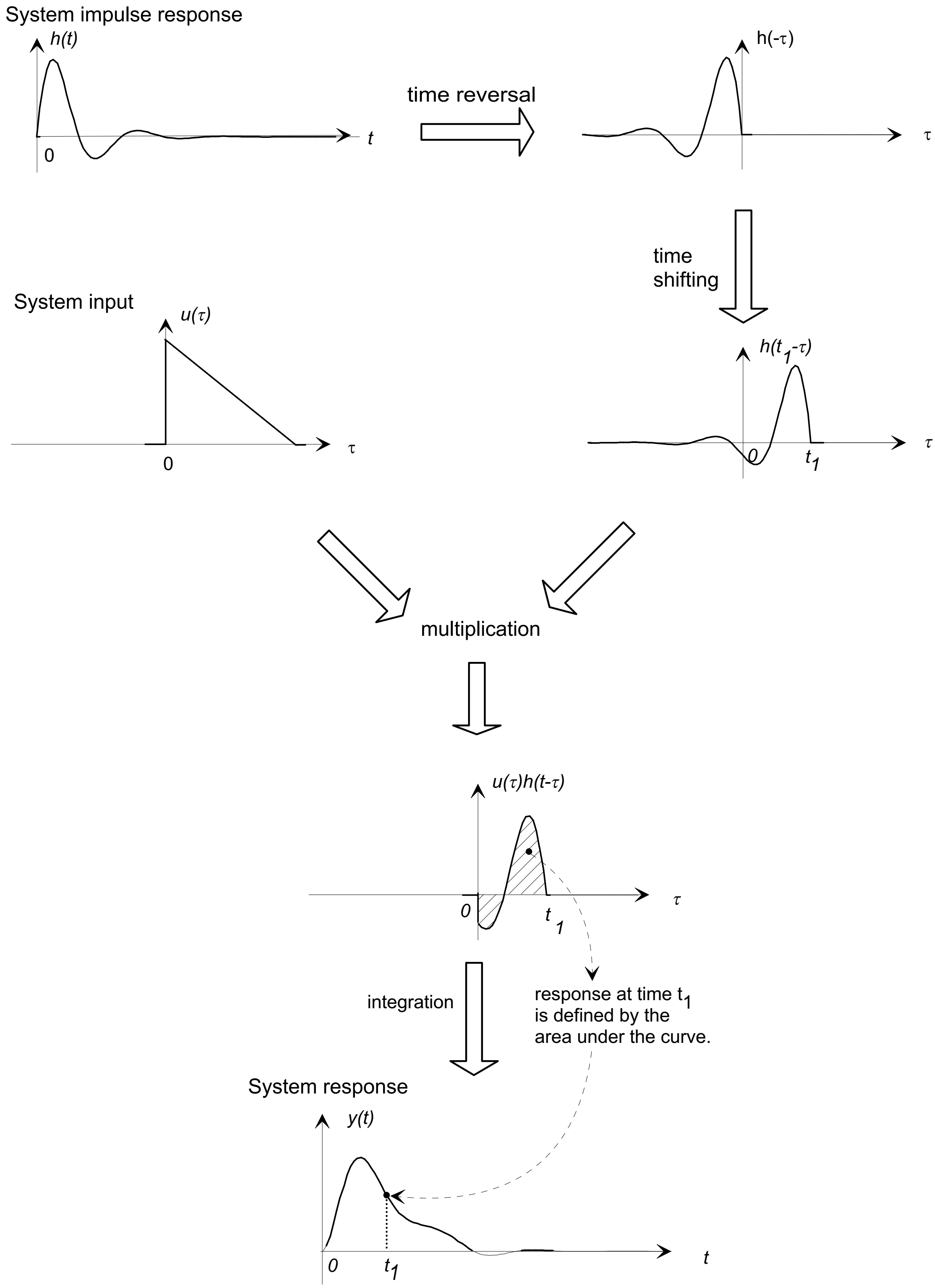
$$y(t) = u(t) \otimes h(t) = \int_{-\infty}^{t} u(\tau)h(t-\tau)d\tau. \tag{2}$$

Equation (**??**) is in the form of a *linear operator*, in that it transforms, or maps, an input function to an output function through a linear operation.



The form of the integral in Eq. (**??**) is difficult to interpret because it contains the term $h(t-\tau)$ in which the variable of integration has been negated. The steps implicitly involved in computing the convolution integral may be demonstrated graphically below. The impulse response $h(\tau)$ is reflected about the origin to create $h(-\tau)$, and then shifted to the right by t to form $h(t-\tau)$. The product $u(t)h(t-\tau)$ is then evaluated and integrated to find the response. This graphical representation is useful for defining the limits necessary in the integration. For example, since for a physical system the impulse response $h(t)$ is zero for all $t < 0$, the reflected and shifted impulse response $h(t-\tau)$ will be zero for all time $\tau > t$. The upper limit in the integral is then at most t. If in addition the input $u(t)$ is time limited, that is $u(t) \equiv 0$ for $t < t_1$ and $t > t_2$, the limits are:

$$y_f(t) = \begin{cases} \displaystyle\int_{t_1}^{t} u(\tau)h(t-\tau)d\tau & \text{for } t < t_2 \\ \displaystyle\int_{t_1}^{t_2} u(\tau)h(t-\tau)d\tau & \text{for } t \geq t_2 \end{cases} \tag{3}$$



See the class handout for further details and examples.

2 Sinusoidal Response of LTI Continuous Systems

Of particular interest is the response of an LTI continuous system to sinusoidal inputs of the form $u(t) = A\sin(\Omega t + \phi)$, where A is the amplitude, Ω is the angular frequency (rad/s), and ϕ is a phase angle (rad). (We note that we can also write $u(t) = A\sin(2\pi F t + \phi)$, where F is the frequency in Hz.)

We begin by noting that a sinusoid may be expressed in terms of complex exponentials through the Euler formulas:

$$\begin{aligned}\sin(\Omega t) &= \frac{1}{2j}\left(e^{j\Omega t} - e^{-j\Omega t}\right)\\ \cos(\Omega t) &= \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right)\end{aligned}$$

and first finding the steady-state solution to inputs of the form $u(t) = e^{j\Omega t}$. Let the LTI system be described by an ODE of the form

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^n} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u.$$

The steady-state response of the system (after all initial condition transients have decayed) may be found using the *method of undetermined coefficients*, in which a form of the solution is assumed and solved for a set of coefficients. In particular, if the input is $u(t) = e^{j\Omega t}$, assume that

$$y(t) = Be^{j\Omega t}.$$

Substitution into the differential equation gives

$$\begin{aligned}&\left(a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \cdots + a_1(j\Omega) + a_0\right)Be^{j\Omega t}\\ &\quad = \left(b_m(j\Omega)^m + b_{n-1}(j\Omega)^{m-1} + \cdots + b_1(j\Omega) + b_0\right)e^{j\Omega t}\end{aligned}$$

and solving for B

$$B = \frac{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \cdots + a_1(j\Omega) + a_0}{b_m(j\Omega)^m + b_{n-1}(j\Omega)^{m-1} + \cdots + b_1(j\Omega) + b_0}$$

so that

$$y(t) = H(j\Omega)e^{j\Omega t}$$

where

$$H(j\Omega) = \frac{N(j\Omega)}{D(j\Omega)} = \frac{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \cdots + a_1(j\Omega) + a_0}{b_m(j\Omega)^m + b_{n-1}(j\Omega)^{m-1} + \cdots + b_1(j\Omega) + b_0}$$

$H(j\Omega)$ is defined to be the *frequency response function*, and $N(j\Omega)$ and $D(j\Omega)$ are the numerator and denominator polynomials respectively. We note the following:

- The output $y(t)$ is simply a (multiplicatively) weighted version of the input.
- $H(j\Omega)$ is a property of the system. It is defined entirely by the describing differential equation.

- $H(j\Omega)$ is, in general, complex. Even powers of n and m in $N(s)$ and $D(s)$ will generate real terms in the polynomials, while odd powers will generate imaginary terms.

$$\begin{aligned} |H(j\Omega)| &= \frac{|N(j\Omega)|}{|D(j\Omega)|} \\ \angle H(j\Omega) &= \angle N(j\Omega) - \angle D(j\Omega) \end{aligned}$$

- $H(-j\Omega) = \overline{H}(j\Omega)$, where $\overline{H}(j\Omega)$ is the complex conjugate.

The response to the real sinusoid

$$u(t) = A\sin(\Omega t + \phi) = \frac{A}{2j}\left(e^{j(\Omega t+\phi)} - e^{-j(\Omega t+\phi)}\right)$$

may be found from the *principle of superposition* by summing the response to each component:

$$\begin{aligned} y(t) &= \frac{A}{2j}\left(H(j\Omega)e^{j(\Omega t+\phi)} - H(-j\Omega)e^{-j(\Omega t+\phi)}\right) \\ &= \frac{A}{2j}\left(H(j\Omega)e^{j(\Omega t+\phi)} - \overline{H(j\Omega)}e^{-j(\Omega t+\phi)}\right) \end{aligned}$$

Combining the real and imaginary parts gives the result

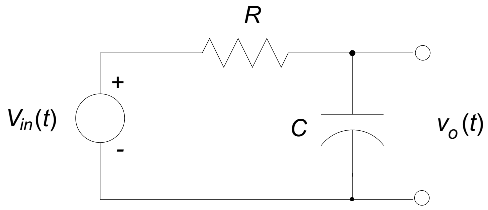
$$y(t) = A\,|H(j\Omega)|\sin(\Omega t + \phi + \angle H(j\Omega))$$

where $|H(j\Omega)|$ is the *magnitude* of the frequency response function, and $\angle H(j\Omega)$ is the *phase* response.

- The response to a real sinusoid is therefore a sinusoid of the same frequency as the input.
- The amplitude of the response at an input frequency of Ω has been modified by a factor $|H(j\Omega)|$. If $|H(j\Omega)| > 1$ the input has been *amplified* by the system, if $|H(j\Omega)| < 1$, the signal has been *attenuated.*
- The system has imposed a frequency dependent phase shift $\angle H(j\Omega)$ on the response.

■ Example 1

A first-order passive RC filter with the following circuit diagram



is described by the differential equation

$$RC\frac{dv_o}{dt} + v_o = V_{in}(t)$$

Find the frequency response function.

By inspection

$$H(j\Omega) = \frac{1}{jRC\Omega + 1}$$

and

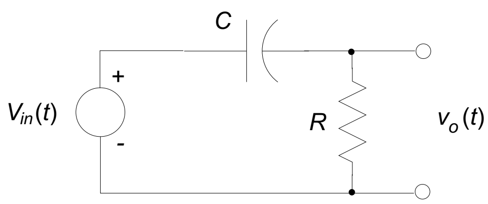
$$\begin{aligned} |H(j\Omega)| &= \frac{|1|}{|1 + jRC\Omega|} = \frac{1}{\sqrt{(RC\Omega)^2 + 1}} \\ \angle H(j\Omega) &= \angle(1) - \angle(1 + jRC\Omega) = 0 - \tan^{-1}(RC\Omega) \end{aligned}$$

Clearly, as $\Omega \to 0$, $|H(j\Omega)| \to 1$, and $\angle H(j\Omega) \to 0$ rad. As $\Omega \to \infty$, $|H(j\Omega)| \to 0$, and $\angle H(j\Omega) \to -\pi/2$ rad (-90°).

This is a *low-pass filter*, in that it passes low frequency sinusoids while attenuating high frequencies.

■ Example 2

A new first-order passive RC filter is formed by exchanging the resistor and capacitor in the previous example:



and is now described by the differential equation

$$RC\frac{dv_o}{dt} + v_o = RC\frac{dV_{in}}{dt}$$

Find the frequency response function.

By inspection

$$H(j\Omega) = \frac{jRC\Omega}{jRC\Omega + 1}$$

and

$$\begin{aligned} |H(j\Omega)| &= \frac{|jRC\Omega|}{|1 + jRC\Omega|} = \frac{RC\Omega}{\sqrt{(RC\Omega)^2 + 1}} \\ \angle H(j\Omega) &= \angle(jRC\Omega) - \angle(1 + jRC\Omega) = \frac{\pi}{2} - \tan^{-1}(RC\Omega) \end{aligned}$$

Clearly, as $\Omega \to 0$, $|H(j\Omega)| \to 0$, and $\angle H(j\Omega) \to \pi/2$ rad (90°). As $\Omega \to \infty$, $|H(j\Omega)| \to 1$, and $\angle H(j\Omega) \to 0$ rad (0°).

This is a *high-pass filter*, in that it attenuates low frequency sinusoids while passing high frequencies.
