2.161 Signal Processing: Continuous and Discrete
Fall 2008

Lecture 8[1]

Reading:

- Class handout: *Introduction to Continuous Time Filter Design.*

- Class Handout: *Introduction to Operational Amplifiers.*

- Class Handout: *Op-Amp Implementation of Analog Filters.*

1 Second-Order Filter Sections

(a) Low-Pass Filter

$$H_{lp}(s) = \frac{a_0}{s^2 + a_1 s + a_0}$$

High Frequency Behavior: The number of poles exceeds the number of zeros ($n - m = 2$) so that
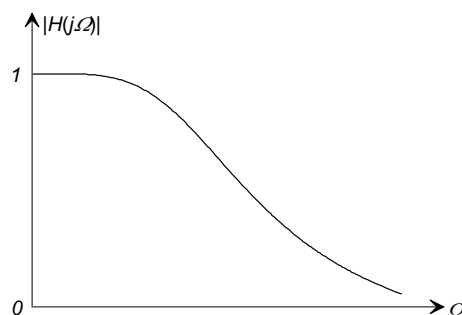
$$\lim_{\Omega \to \infty} |H(j\Omega)| = 0.$$

and the high frequency asymptotic slope is -40dB/decade.

Low Frequency Behavior:

$$\lim_{\Omega \to 0} |H(j\Omega)| = 1$$

Mid Frequency Behavior: The response in the region $\Omega \approx \sqrt{a_0}$ is determined by the systems damping ratio ζ, and will exhibit a resonant peak if $\zeta < 0.707$.

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(b) High-Pass Filter

$$H_{hp}(s) = \frac{s^2}{s^2 + a_1 s + a_0}$$

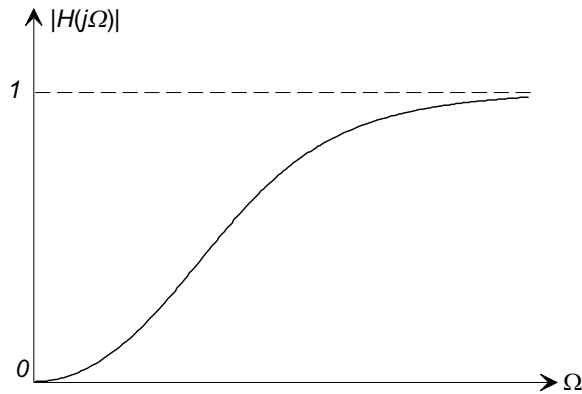**High Frequency Behavior:** The number of poles equals the number of zeros ($n = m$) so that

$$\lim_{\Omega \to \infty} |H(j\Omega)| = 1.$$

Low Frequency Behavior: There are a pair of zeros at the origin so that

$$\lim_{\Omega \to 0} |H(j\Omega)| = 0$$

and the low frequency asymptotic slope is +40dB/decade.

Mid Frequency Behavior: The response in the region $\Omega \approx \sqrt{a_0}$ is determined by the systems damping ratio ζ, and will exhibit a resonant peak if $\zeta < 0.707$.

(c) Band-Pass Filter

$$H_{bp}(s) = \frac{a_1 s}{s^2 + a_1 s + a_0}$$

High Frequency Behavior: The number of poles exceeds the number of zeros ($n - m = 1$) so that

$$\lim_{\Omega \to \infty} |H(j\Omega)| = 0.$$

and the high frequency asymptotic slope is -20dB/decade.

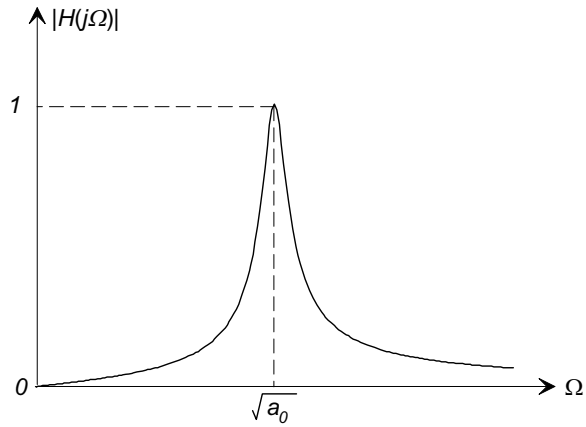**Low Frequency Behavior:** There is a single of zero at the origin so that

$$\lim_{\Omega \to 0} |H(j\Omega)| = 0$$

and the low frequency asymptotic slope is +20dB/decade.

Mid Frequency Behavior: When $s = j\sqrt{a_0}$,

$$H(j\sqrt{a_0}) = 1$$

which defines the passband center frequency.

(d) Band-Stop Filter

$$H_{bs}(s) = \frac{s^2 + a_0}{s^2 + a_1 s + a_0}$$

High Frequency Behavior: The number of poles equals the number of zeros ($n = m = 2$) so that

$$\lim_{\Omega \to \infty} |H(j\Omega)| = 1.$$

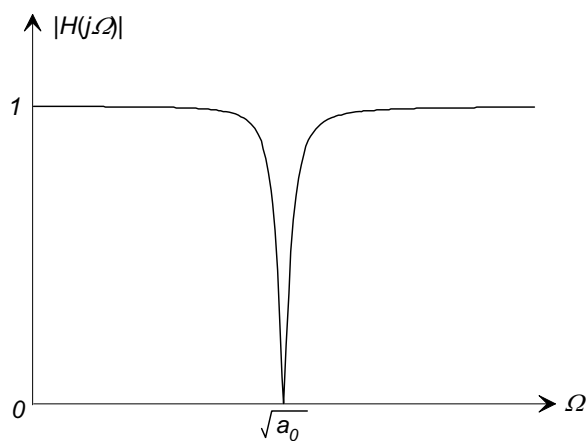**Low Frequency Behavior:** There are no zeros at the origin and

$$\lim_{\Omega \to 0} |H(j\Omega)| = 1$$

Mid Frequency Behavior: There are a pair of imaginary zeros at $s = \pm j\sqrt{a_0}$ forcing the response magnitude to zero at a frequency $\Omega = \sqrt{a_0}$.

$$|H(j\sqrt{a_0})| = 0$$

which defines the band rejection (notch) center frequency.

2 Transformation of Low-Pass Filters to other Classes

Filter specification tolerance bands for high-pass, band-pass and band-stop filters are shown The most common procedure for the design of these filters is to design a *prototype low-pass filter* using the methods described above, and then to *transform* the low-pass filter to the desired form by a substitution in the transfer function, that is we substitute a function $g(s)$ for s in the low-pass transfer function $H_{lp}(s)$, so that the new transfer function is $H'(s) = H_{lp}(g(s))$. The effect is to modify the filter poles and zeros to produce the desired frequency response characteristic.

(a) High-pass

(b) Band-pass
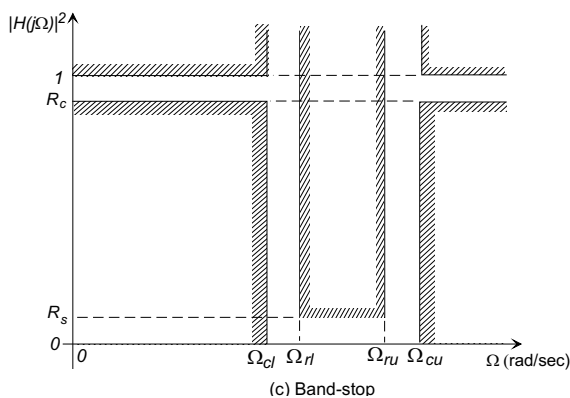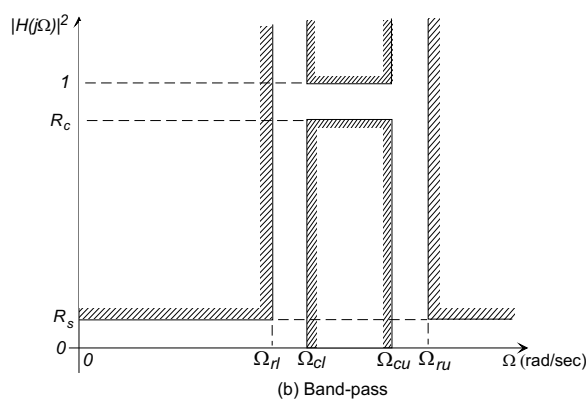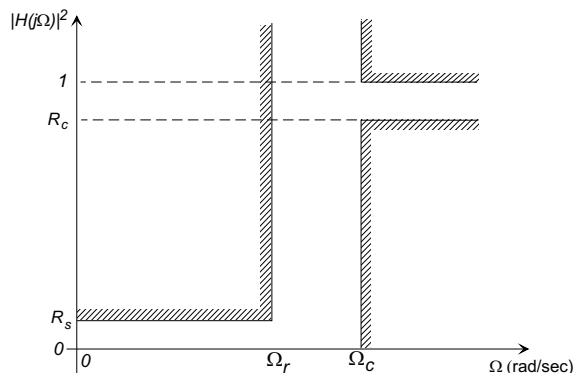
(c) Band-stop

The critical frequencies used in the design are as shown above. For band-pass and band-stop filters it is convenient to define a center frequency Ω_o as the geometric mean of the pass-pand

edges, and a bandwidth $\Delta\Omega$:

$$\Omega_o = \sqrt{\Omega_{cu}\Omega_{cl}}$$
$$\Delta_\Omega = \Omega_{cu} - \Omega_{cl}.$$

The transformation formulas for a low-pass filter with cut-off frequency Ω_c are given below

Low-pass $(\Omega_{c_1}) \rightarrow$ Low-pass (Ω_{c_2})	$g(s) = \dfrac{\Omega_{c_1}}{\Omega_{c_2}}s$
Low-pass $(\Omega_c) \rightarrow$ High-pass (Ω_c)	$g(s) = \dfrac{\Omega_c^2}{s}$
Low-pass $(\Omega_c = \Delta_\Omega) \rightarrow$ Band-pass $(\Omega_{cl}, \Omega_{cu})$	$g(s) = \dfrac{s^2 + \Omega_o^2}{s}$
Low-pass $(\Omega_c = \Delta_\Omega) \rightarrow$ Band-stop $(\Omega_{cl}, \Omega_{cu})$	$g(s) = \dfrac{s\Omega_c^2}{s^2 + \Omega_o^2}$

The band-pass and band-stop transformations both double the order of the filter, since s^2 is involved it the transformation. The low-pass filter is designed to have a cut-off frequency $\Omega_c = \Omega_{cu} - \Omega_{cl}$.

The above transformations will create an ideal gain characteristic from an ideal low-pass filter. For practical filters, however, the "skirts" of the pass-bands will be a warped representation of the low-pass prototype filter. This does not usually cause problems.

■ Example 1

Transform the first-order low-pass filter

$$H_{lp}(s) = \frac{\Omega_c}{s + \Omega_c}$$

to a high-pass filter $H_{hp}(s)$.

Using the transformation $g(s) = \Omega_c^2/s$

$$H_{hp}(s) = \frac{\Omega_c}{\left(\frac{\Omega_c^2}{s}\right) + \Omega_c} = \frac{s}{s + \Omega_c}$$

■ Example 2

Transform the second-order low-pass Butterworth filter

$$H_{lp}(s) = \frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

to a high-pass filter $H_{hp}(s)$.

Using the transformation $g(s) = \Omega_c^2/s$

$$H_{hp}(s) = \frac{\Omega_c^2}{\left(\frac{\Omega_c^2}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c^2}{s}\right) + \Omega_c^2} = \frac{s^2}{s^2 + \sqrt{2}s + \Omega_c}$$

■ Example 3

Design a second-order bandpass filter with center frequency Ω_o and bandwidth Δ_Ω.

Step 1: Design a first-order prototype low-pass filter with cut-off frequency Δ_Ω:

$$H_{lp}(s) = \frac{\Delta_\Omega}{s + \Delta_\Omega}$$

Step 2: Transform the prototype using

$$g(s) = \frac{s^2 + \Omega_o^2}{s}$$

so that

$$H(s) = \frac{\Delta_\Omega}{\left(\frac{s^2 + \Omega_o^2}{s}\right) + \Delta_\Omega} = \frac{\Delta_\Omega s}{s^2 + \Delta_\Omega s + \Omega_o^2}$$

■ Example 4

Design a second-order band-stop filter with center frequency Ω_o and notch-width Δ_Ω.

Step 1: Design a first-order prototype low-pass filter with cut-off frequency Δ_Ω:

$$H_{lp}(s) = \frac{\Delta_\Omega}{s + \Delta_\Omega}$$

Step 2: Transform the prototype using

$$g(s) = \frac{s\Delta_\Omega^2}{s^2 + \Omega_o^2}$$

so that

$$H(s) = \frac{\Delta_\Omega}{\left(\frac{s\Delta_\Omega^2}{s^2 + \Omega_o^2}\right) + \Delta_\Omega} = \frac{s^2 + \Omega_o^2}{s^2 + \Delta_\Omega s + \Omega_o^2}$$

3   State-Variable Active Filters

Practical realizations of analog filters are usually based on factoring the transfer function into cascaded second-order sections, each based on a complex conjugate pole-pair or a pair of real poles, and a first-order section if the order is odd. Any zeros in the system may be distributed among the second- and first-order sections. Each first- and second-order section is then implemented by an active filter and connected in series. For example the third-order Butterworth high-pass filter

$$H(s) = \frac{s^3}{s^3 + 40s^2 + 800s + 8000}$$

would be implemented as

$$H(s) = \frac{s^2}{s^2 + 20s + 400} \times \frac{s}{s + 20}$$

as shown below:

(a)   (b)
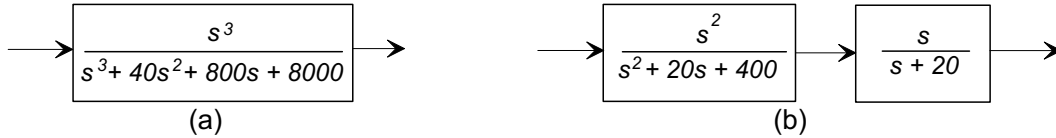
The design of each low-order block can be handled independently.

The *state-variable* filter design method is based on the block diagram representation used in the so-called *phase-variable* description of linear systems that uses the outputs of a chain of cascaded integrators as state variables. Consider a second-order filter block with a transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

and split $H(s)$ into two sub-blocks representing the denominator and numerator by introducing an intermediate variable x and rewrite

$$
\begin{aligned}
H_1(s) &= \frac{X(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_0} \\
H_2(s) &= \frac{Y(s)}{X(s)} = b_2 s^2 + b_1 s + b_0 \\
\text{so that} \quad H(s) &= H_2(s)H_1(s).
\end{aligned}
$$

The differential equations corresponding to $H_1(s)$ and $H_2(s)$ are

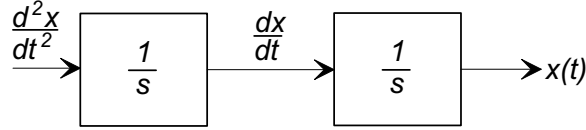$$\frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = u$$

and
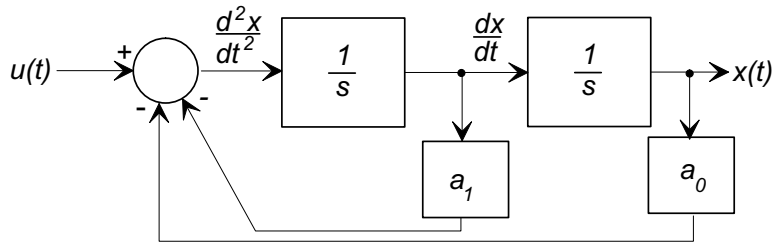
$$y = b_2 \frac{d^2 x}{dt^2} + b_1 \frac{dx}{dt} + b_0 x.$$

The first equation may be rewritten explicitly in terms of the highest derivative

$$\boxed{\frac{d^2x}{dt^2} = -a_1\frac{dx}{dt} - a_0x + u.}$$

(1)

Consider a pair of cascaded analog integrators with the output defined as $x(t)$ so that the derivatives of $x(t)$ appear as inputs to the integrators:


Note that Eq. (1) gives an explicit expression for the input to the first block in terms of the outputs of the two integrators and the system input, and therefore generates the block diagram for $H_1(s)$ shown below:

The equation

$$y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0x.$$

shows that the output $y(t)$ is a weighted sum of $x(t)$ and its derivatives, leading to the complete second-order state variable filter block diagram:
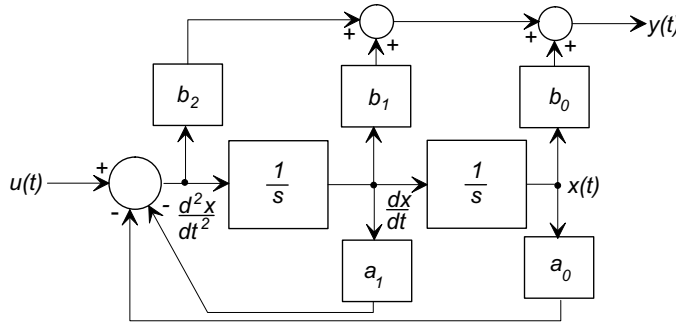
This basic structure may be used to realize the four basic filter types by appropriate choice of the numerator.

$$H_{lp}(s) = \frac{Y_1(s)}{U(s)} = \frac{a_0}{s^2 + a_1s + a_0} \qquad \text{a unity gain low-pass filter} \qquad (2)$$

$$H_{bp}(s) = \frac{Y_2(s)}{U(s)} = \frac{a_1s}{s^2 + a_1s + a_0} \qquad \text{a unity gain band-pass filter} \qquad (3)$$

$$H_{hp}(s) = \frac{Y_3(s)}{U(s)} = \frac{s^2}{s^2 + a_1s + a_0} \qquad \text{a unity gain high-pass filter} \qquad (4)$$

$$H_{bs}(s) = \frac{Y_4(s)}{U(s)} = \frac{s^2 + a_0}{s^2 + a_1s + a_0} \qquad \text{a unity gain band-stop filter} \qquad (5)$$