2.161 Signal Processing: Continuous and Discrete
Fall 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

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Fall Term 2008

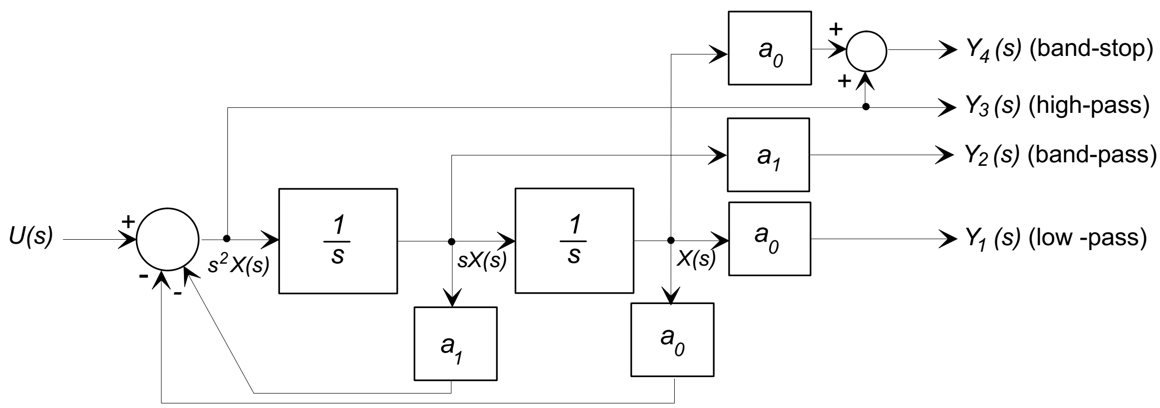
Lecture 9[1]

Reading:

- Class Handout: *Introduction to the Operational Amplifier*
- Class Handout: *Op-amp Implementation of Analog Filters*

1 Operational-Amplifier Based State-Variable Filters

We saw in Lecture 8 that second-order filters may be implemented using the block diagram structure

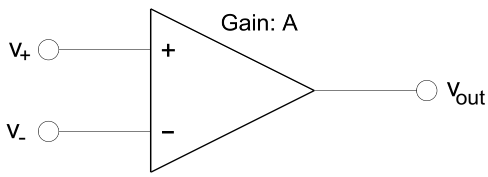


and that a high-order filter may be implemented by cascading second-order blocks, and possibly a first-order block (if the filter order is odd).

We now look into a method for implementing this filter structure using operational amplifiers.

1.1 The Operational Amplifier

What is an operational amplifier? It is simply a very high gain electronic amplifier, with a pair of differential inputs. Its functionality comes about through the use of *feedback* around the amplifier, as we show below.



The op-amp has the following characteristics:

- It is basically a "three terminal" amplifier, with two inputs and an output. It is a *differential* amplifier, that is the output is proportional to the *difference* in the voltages applied to the two inputs, with very high gain A,

$$v_{out} = A(v_+ - v_-)$$

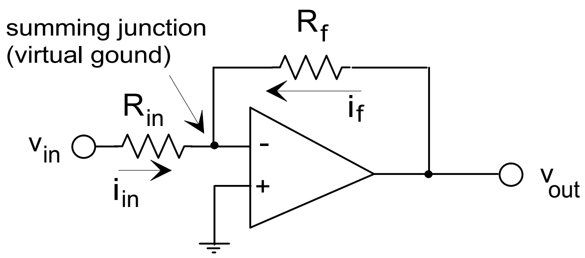
  where A is typically $10^4 - 10^5$, and the two inputs are known as the *non-inverting* (v_+) and *inverting* (v_-) inputs respectively. In the ideal op-amp we assume that the gain A is infinite.

- In an ideal op-amp no current flows into either input, that is they are voltage-controlled and have infinite input resistance. In a practical op-amp the input current is in the order of pico-amps (10^{-12}) amp, or less.

- The output acts as a voltage source, that is it can be modeled as a Thevenin source with a very low source resistance.

The following are some common op-amp circuit configurations that are applicable to the active filter design method described here. (See the class handout for other common configurations).

The Inverting Amplifier:



In the configuration shown above we note

- Because the gain A is very large, the voltage at the node designated *summing junction* is very small, and we approximate it as $v_- = 0$ — the so-called virtual ground assumption.

- We assume that the current i_- into the inverting input is zero.

Applying Kirchoff's Current law at the summing junction we have

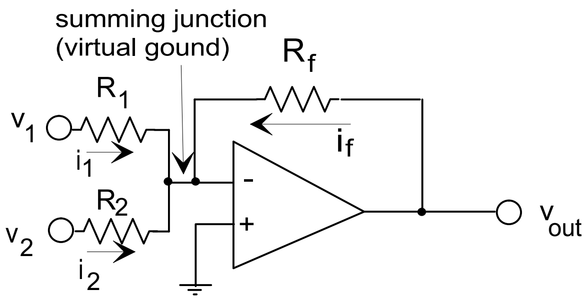
$$i_1 + i_f = \frac{v_{in}}{R_1} + \frac{v_o}{R_f} = 0$$

from which

$$\boxed{v_{out} = -\frac{R_f}{R_{in}} v_{in}}$$

The voltage gain is therefore defined by the ratio of the two resistors. The term *inverting* amplifier comes about because of the sign change.

The Inverting Summer: The inverting amplifier may be extended to include multiple inputs:



As before we assume that the inverting input is at a virtual ground ($v_- \approx 0$) and apply Kirchoff's current law at the summing junction

$$i_1 + i_2 + i_f = \frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{V_{out}}{R_f} = 0$$

or

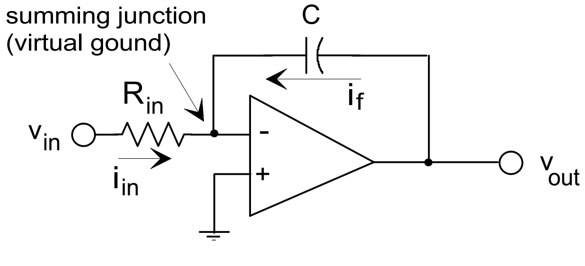
$$\boxed{v_{out} = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2\right)}$$

which is a weighted sum of the inputs.

The summer may be extended to include several inputs by simply adding additional input resistors R_i, in which case

$$\boxed{v_{out} = -\sum_{i=1}^{n} \frac{R_f}{R_i} v_i}$$

The Integrator: If the feedback resistor in the inverting amplifier is replaced by a capacitor C the amplifier becomes an integrator.



At the summing junction we apply Kirchoff's current law as before but the feedback current is now defined by the elemental relationship for the capacitor:

$$i_{in} + i_f = \frac{v_{in}}{R_{in}} + C\frac{dv_{out}}{dt} = 0$$

Then

$$\frac{dv_{out}}{dt} = -\frac{1}{R_{in}C}v_{in}$$

or

$$\boxed{v_{out} = -\frac{1}{R_{in}C}\int_0^t v_{in}dt + v_{out}(0)}$$

As above, the integrator may be extended to a summing configuration by simply adding extra input resistors:

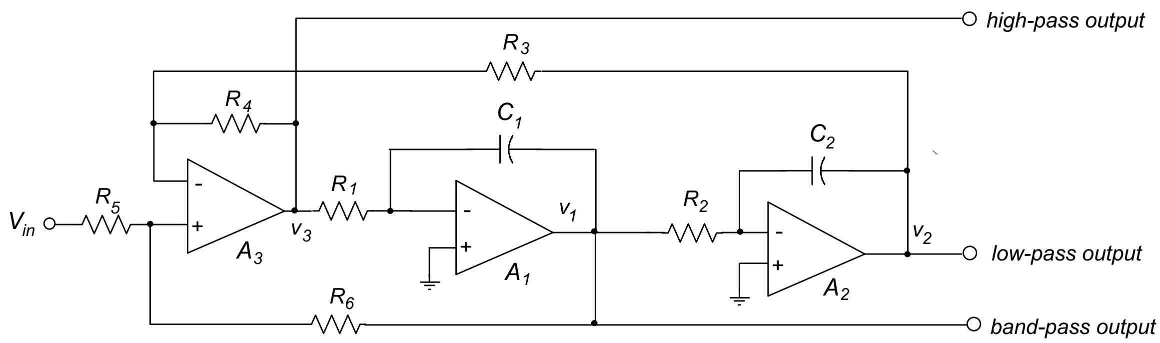
$$v_{out} = -\frac{1}{C}\int_0^t \left(\sum_{i=1}^{n} \frac{v_i}{R_i}\right) dt + v_{out}(0)$$

and if all input resistors have the same value R

$$v_{out} = -\frac{1}{RC}\int_0^t \left(\sum_{i=1}^{n} v_i\right) dt + v_{out}(0)$$

1.2 A Three Op-Amp Second-Order State Variable Filter

A configuration using three op-amps to implement low-pass, high-pass, and bandbass filters directly is shown below:



Amplifiers A_1 and A_2 are integrators with transfer functions

$$H_1(s) = -\left(\frac{1}{R_1C_1}\right)\frac{1}{s} \quad \text{and} \quad H_2(s) = -\left(\frac{1}{R_2C_2}\right)\frac{1}{s}.$$

Let $\tau_1 = R_1C_1$ and $\tau_2 = R_2C_2$. Because of the gain factors in the integrators and the sign inversions we have

$$v_1(t) = -\tau_2\frac{dv_2}{dt} \quad \text{and} \quad v_3(t) = \tau_1\tau_2\frac{d^2v_2}{dt^2}.$$

Amplifier A_3 is the summer. However, because of the sign inversions in the op-amp circuits we cannot use the elementary summer configuration described above. Applying Kirchoff's Current Law at the non-inverting and inverting inputs of A_3 gives

$$\frac{V_{in} - v_+}{R_5} + \frac{v_1 - v_+}{R_6} = 0 \quad \text{and} \quad \frac{v_3 - v_-}{R_4} + \frac{v_2 - v_-}{R_1} = 0.$$

Using the infinite gain approximation for the op-amp, we set $v_- = v_+$ and

$$\frac{R_3}{R_3+R_4}v_3 - \frac{R_5}{R_5+R_6}v_1 + \frac{R_4}{R_3+R_4}v_2 = \frac{R_6}{R_5+R_6}V_{in},$$

and substituting for v_1 and v_3 we generate a differential equation in v_2

$$\boxed{\frac{d^2v_2}{dt^2} + \left(\frac{1+R_4/R_3}{\tau_1(1+R_6/R_5)}\right)\frac{dv_2}{dt} + \left(\frac{R_4}{R_3}\frac{1}{\tau_1\tau_2}\right)v_2 = \left(\frac{1+R_4/R_3}{\tau_1\tau_2\,(1+R_5/R_6)}\right)V_{in}}$$

which corresponds to a low-pass transfer function with

$$H(s) = \frac{K_{lp}a_0}{s^2+a_1s+a_0}$$

where

$$\begin{aligned} a_0 &= \left(\frac{R_4}{R_3}\right)\frac{1}{\tau_1\tau_2} \\ a_1 &= \left(\frac{1+R_4/R_3}{1+R_6/R_5}\right)\frac{1}{\tau_1} \\ K_{lp} &= \frac{1+R_3/R_4}{1+R_5/R_6} \end{aligned}$$

A Band-Pass Filter: Selection of the output as the output of integrator A_1 generates the transfer function

$$H_{bp}(s) = -\tau_1 s H_{lp}(s) = \frac{-K_{bp}a_1 s}{s^2+a_1s+a_0}$$

where

$$K_{bp} = \frac{R_6}{R_5}$$

A High-Pass Filter: Selection of the output as the output of the summer A_3 generates the transfer function

$$H_{hp}(s) = \tau_1\tau_2 s^2 H_{lp}(s) = \frac{K_{hp}s^2}{s^2+a_1s+a_0}$$

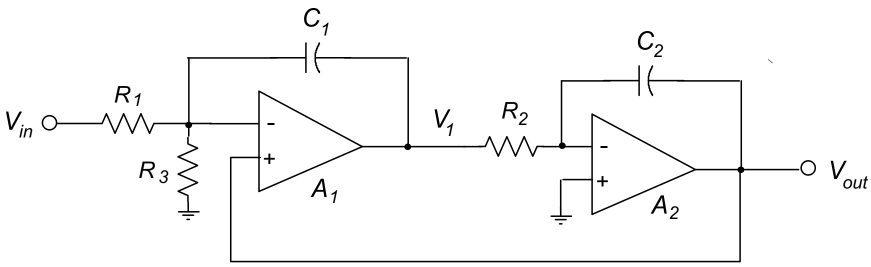
where

$$K_{hp} = \frac{1+R_4/R_3}{1+R_5/R_6}$$

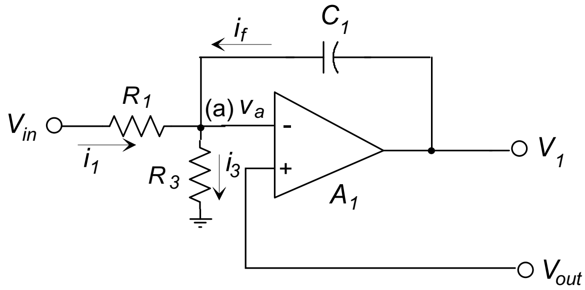
A Band-Stop Filter: The band-stop configuration may be implemented with an additional summer to add the outputs of amplifiers A_2 and A (with appropriate weights).

1.3 A Simplified Two Op-amp Based State-variable Filter:

If the required filter does not require a high-pass action (that is, access to the output of the summer A_1 above) the summing operation may be included at the input of the first integrator, leading to a simplified circuit using only two op-amps shown below:



Consider the input stage:



With the infinite gain assumption for the op-amps, that is $V_- = V_+$, and with the assumption that no current flows in either input, we can apply Kirchoff's Current Law (KCL) at the node designated (a) above:

$$i_1 + i_f - i_3 = (V_{in} - v_a)\frac{1}{R_1} + sC_1(v_1 - v_a) - v_a\frac{1}{R_3} = 0$$

Assuming $v_a = V_{out}$, and realizing that the second stage is a classical op-amp integrator with transfer function

$$\frac{V_{out}(s)}{v_1(s)} = -\frac{1}{R_2C_2s}$$

$$(V_{in} - V_{out})\frac{1}{R_1} + sC_1(-R_2C_2sV_{out} - V_{out}) - V_{out}\frac{1}{R_3} = 0$$

which may be rearranged to give the second-order transfer function

$$\boxed{\frac{V_{out}(s)}{V_{in}(s)} = \frac{1/\tau_1\tau_2}{s^2 + (1/\tau_2)s + (1 + R_1/R_3)/\tau_1\tau_2}}$$

which is of the form

$$H_{lp}(s) = \frac{K_{lp}a_0}{s^2 + a_1s + a_0}$$

where

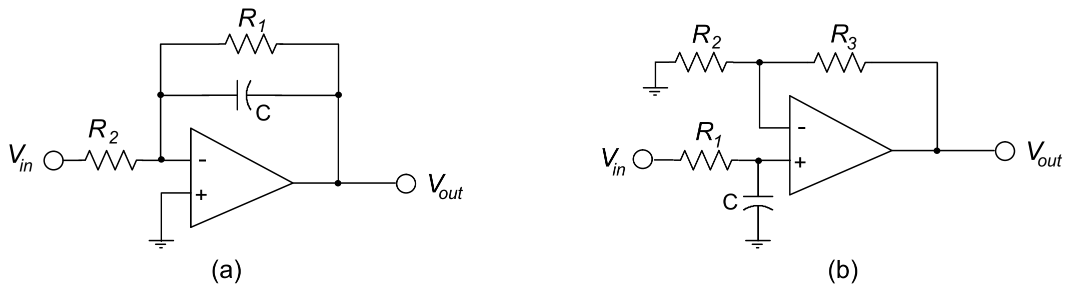
$$\begin{aligned} a_0 &= (1 + R_1/R_3)\frac{1}{\tau_1\tau_2} \\ a_1 &= \frac{1}{\tau_2} \\ K_{lp} &= \frac{1}{1 + R_1/R_3} \end{aligned}$$

1.4 First-Order Filter Sections:

Single pole low-pass filter sections with a transfer function of the form

$$H(s) = \frac{K\Omega_0}{s + \Omega_0}$$

may be implemented in either an inverting or non-inverting configuration as shown in Fig. 11.



The inverting configuration (a) has transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f}{Z_{in}} = -\left(\frac{R_1}{R_2}\right)\frac{1/R_1C}{s + 1/R_1C}$$

where $\Omega_0 = 1/R_1C$ and $K = -R_1/R_2$.

The non-inverting configuration (b) is a first-order R-C lag circuit buffered by a non-inverting (high input impedance) amplifier (see the class handout) with a gain $K = 1 + R_3/R_2$. Its transfer function is

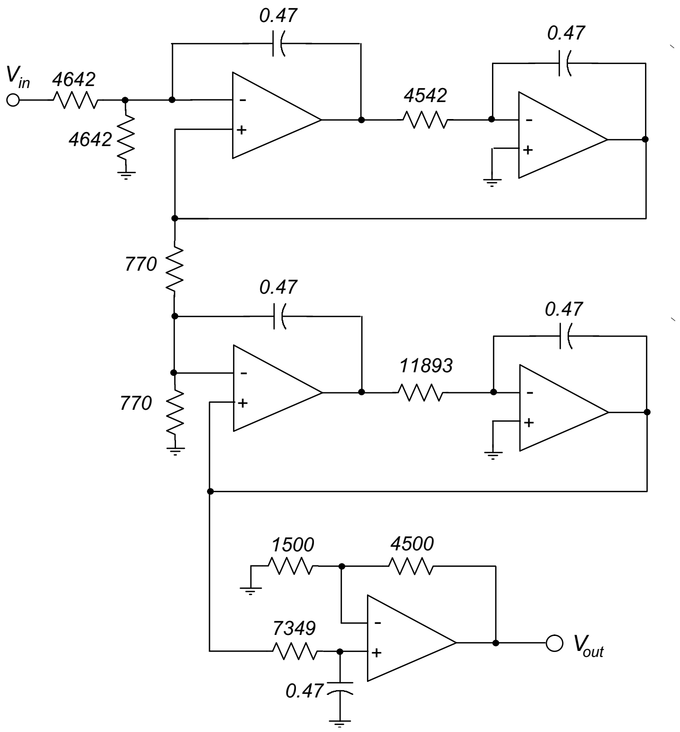
$$\frac{V_{out}(s)}{V_{in}(s)} = \left(1 + \frac{R_3}{R_2}\right)\frac{1/R_1C}{s + 1/R_1C}.$$

Classroom Demonstration

Example 2 in the class handout "Op-Amp Implementation of Analog Filters" describes a state-variable design for a 5th-order Chebyshev Type I low-pass filter with $\Omega_c = 1000$ rad/s and 1dB ripple in the passband.
The transfer function is

$$\begin{aligned} H(s) &= \frac{122828246505000}{(s^2+468.4s+429300)(s^2+178.9s+988300)(s+289.5)} \\ &= \frac{429300}{s^2+468.4s+429300} \times \frac{988300}{s^2+178.9s+988300} \times \frac{289.5}{s+289.5} \end{aligned}$$

which is implemented in the handout as a pair of second-order two-op-amp sections followed by a first-order block:



This filter was constructed on a bread-board using 741 op-amps, and was demonstrated to the class, driven by a sinusoidal function generator and with an oscilloscope to display the output. The demonstration included showing (1) the approximately 10% ripple in the passband, and (2) the rapid attenuation of inputs with frequency above 157 Hz (1000 rad/s).

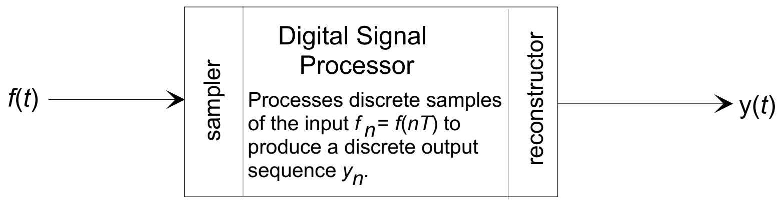
2 Introduction to Discrete-Time Signal Processing

Consider a continuous function $f(t)$ that is limited in extent, $T_1 \leq t < T_2$. In order to process this function in the computer it must be *sampled* and represented by a finite set of numbers. The most common sampling scheme is to use a fixed sampling interval ΔT and to form a sequence of length N: $\{f_n\}$ $(n = 0 \ldots N-1)$, where

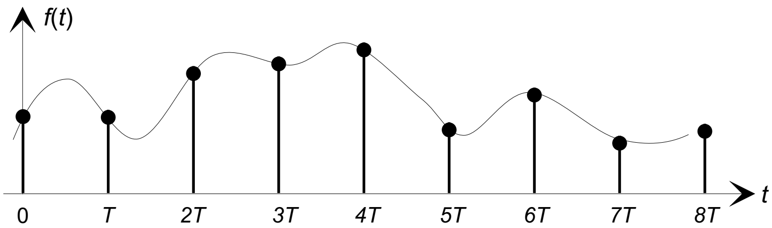
$$f_n = f(T_1 + n\Delta T).$$

In subsequent processing the function $f(t)$ is represented by the finite sequence $\{f_n\}$ and the sampling interval ΔT.

In practice, sampling occurs in the time domain by the use of an analog-digital (A/D) converter.



(i) The sampler (A/D converter) records the signal value at discrete times $n\Delta T$ to produce a sequence of samples $\{f_n\}$ where $f_n = f(n\Delta T)$ (ΔT is the *sampling interval.*



(ii) At each interval, the output sample y_n is computed, based on the history of the input and output, for example

$$y_n = \frac{1}{3}(f_n + f_{n-1} + f_{n-2})$$

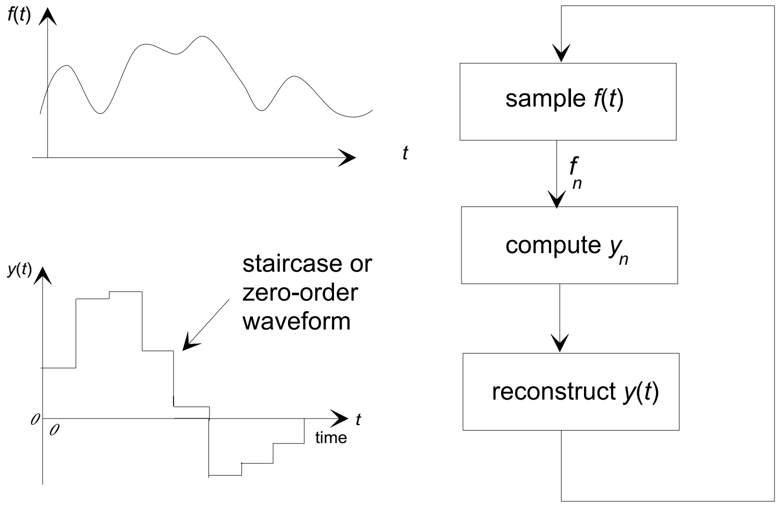
3-point moving average filter, and

$$y_n = 0.8y_{n-1} + 0.2f_n$$

is a simple recursive first-order low-pass digital filter. Notice that they are *algorithms.*

(iii) The reconstructor takes each output sample and creates a continuous waveform.

In real-time signal processing the system operates in an infinite loop:



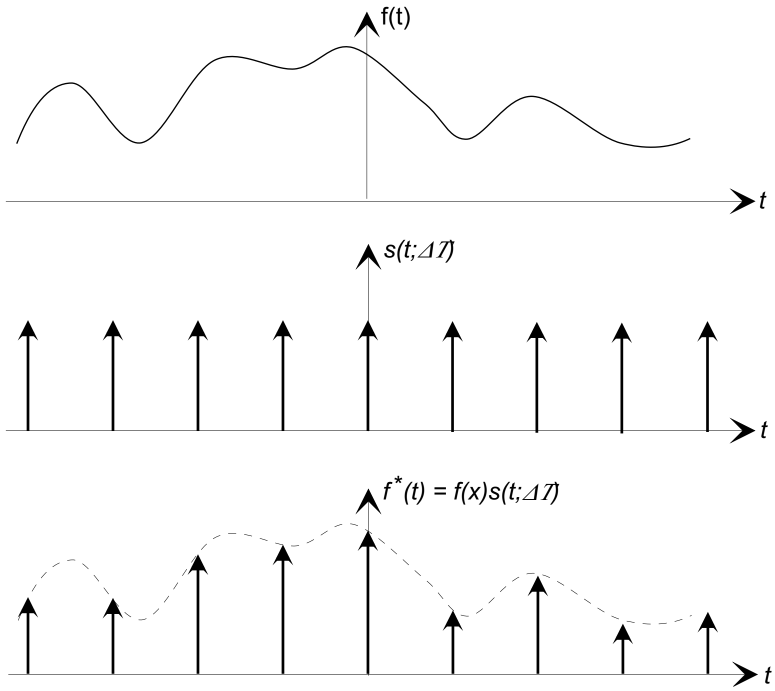
2.1 Sampling

The mathematical operation of sampling (not to be confused with the operation of an analog-digital converter) is most commonly described as a *multiplicative* operation, in which $f(t)$ is multiplied by a *Dirac comb* sampling function $s(t;\Delta T)$, consisting of a set of delayed Dirac delta functions:

$$s(t;\Delta T) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T).$$

We denote the sampled waveform $f^{\star}(t)$ as

$$f^{\star}(t) = s(t;\Delta T)f(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$



Note that $f^\star(t)$ is a set of delayed and weighted delta functions, and that the waveform must be interpreted in the *distribution* sense by the strength (or area) of each component impulse. The implied process to produce the discrete sample sequence $\{f_n\}$ is by integration across each impulse, that is

$$f_n = \int_{n\Delta T^-}^{n\Delta T^+} f^\star(t)dt = \int_{n\Delta T^-}^{n\Delta T^+} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)dt$$

or

$$f_n = f(n\Delta T)$$

by the sifting property of $\delta(t)$.

2.2 The Spectrum of the Sampled Waveform $f^\star(t)$:

Notice that sampling comb function $s(t;\Delta T)$ is periodic and is therefore described by a Fourier series:

$$s(t;\Delta T) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_0 t}$$

where all the Fourier coefficients are equal to $(1/\Delta T)$, and where $\Omega_0 = 2\pi/\Delta T$ is the fundamental angular frequency. Using this form, the spectrum of the sampled waveform $f^\star(t)$ may be written

$$\begin{aligned} F^\star(j\Omega) &= \int_{-\infty}^{\infty} f^\star(t)\, \mathrm{e}^{-j\Omega t}\, \mathrm{d}t = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{jn\Omega_0 t}\, \mathrm{e}^{-j\Omega t}\, \mathrm{d}t \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(j\left(\Omega - n\Omega_0\right)\right) \\ &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(j\left(\Omega - \frac{2\pi n}{\Delta T}\right)\right) \end{aligned}$$

The Fourier transform of a sampled function $f^\star(t)$ is periodic in the frequency domain with period $\Omega_0 = 2\pi/\Delta T$, and is a *superposition* of an infinite number of shifted replicas of the Fourier transform, $F(j\Omega)$, of the original function scaled by a factor of $1/\Delta T$.