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1)

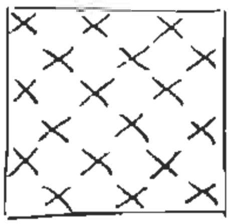
	1	2	3	4	5	6 ← SECOND DIE
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

↑ FIRST DIE

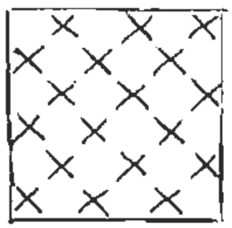
⇐ VALUES OF THE SUM OF THE ROLL

PROBABILITY OF EACH COMBINATION EQUALS $\frac{1}{36}$.

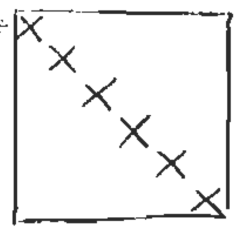
DEFINE EVENTS AS FOLLOWS:

A = 

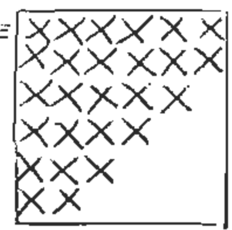
$P[A] = \frac{18}{36} = \frac{1}{2}$

B = 

$P[B] = \frac{18}{36} = \frac{1}{2}$

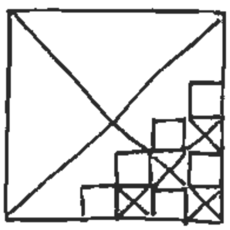
C = 

$P[C] = \frac{6}{36} = \frac{1}{6}$

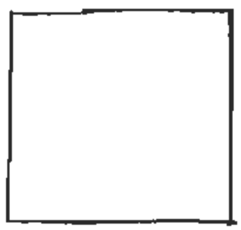
D = 

$P[D] = \frac{26}{36} = \frac{13}{18}$

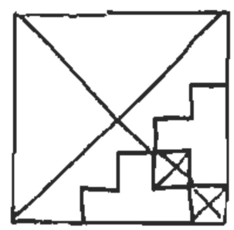
a) $P(A \cup D) = \frac{30}{36} = \boxed{\frac{5}{6}}$



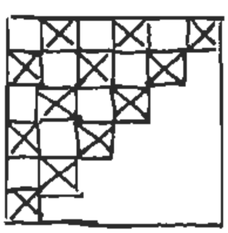
b) $P(B \cap C) = \boxed{\emptyset}$



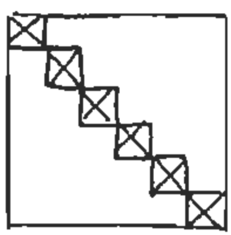
c) $P(C \cup D) = \frac{28}{36} = \boxed{\frac{7}{9}}$



d) $P(B \cap D) = \frac{12}{36} = \boxed{\frac{1}{3}}$



e) $P(A \cap C) = \frac{6}{36} = \boxed{\frac{1}{6}}$



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2) P[FULL HOUSE]

WHERE FULL HOUSE = A FIVE CARD HAND IN WHICH THERE IS A 3-OF-A-KIND AND A 2-OF-A-KIND

$$P[\text{FULL HOUSE}] = \frac{\#\text{ OF POSSIBLE FULL HOUSES}}{\text{TOTAL }\#\text{ OF POSSIBLE 5CARD HANDS}}$$

FIRST LOOK AT THE TOTAL # OF POSSIBLE HANDS:

CHOOSE 5 CARDS OUT OF 52 TOTAL = $\binom{52}{5} = \frac{52!}{47!\,5!}$

$$\binom{52}{5} = \frac{52!}{47!\,5!} = \frac{52\cdot 51\cdot \overset{10}{\cancel{50}}\cdot 49\cdot \overset{2}{\cancel{48}}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 52\cdot 51\cdot 49\cdot 10\cdot 2$$

NOW LOOK AT POSSIBLE FULL HOUSES:

THERE ARE 13 DIFFERENT FACE VALUES (e.g. A, 2, 3 etc.) WHICH YOU CAN CHOOSE FOR THE 3 OF A KIND $\binom{13}{1}$

ONCE YOU'VE CHOSEN A FACE VALUE, THERE ARE FOUR POSSIBLE SUITS (♡, ◇, ♧, ♤) SO THERE ARE $\binom{4}{3}$ POSSIBLE 3-OF-A-KINDS FOR EACH FACE VALUE

FOR EACH CHOICE OF FACE VALUE FOR THE 3-OF-A-KIND, THERE ARE 12 POSSIBLE CHOICES FOR THE 2-OF-A-KIND $\binom{12}{1}$

ONCE AGAIN, THERE ARE FOUR SUITS TO CHOOSE THE 2 CARDS FROM SO THERE ARE $\binom{4}{2}$ POSSIBLE COMBINATIONS

THEREFORE THERE ARE $\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{1}\binom{4}{2}$ POSSIBLE FULL HOUSES

$$\binom{13}{1} = \frac{13!}{12!\,1!} = 13 \qquad \binom{4}{3} = \frac{4!}{3!\,1!} = 4 \qquad \binom{12}{1} = \frac{12!}{11!\,1!} = 12 \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

$$\therefore P[\text{FULL HOUSE}] = \frac{\cancel{13}\cdot\cancel{4}\cdot\cancel{12}\cdot\cancel{6}}{\cancel{52}\cdot\underset{17}{\cancel{51}}\cdot 49\cdot\underset{5}{\cancel{10}}\cdot 8} = \frac{6}{5\cdot 17\cdot 49} = \boxed{\frac{6}{4165}}$$

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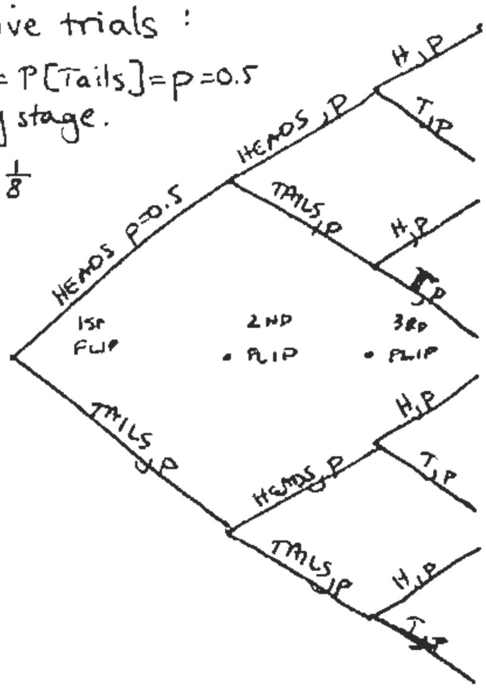
3) $X = \{\text{sum of 3 flips of a fair coin where heads} = 1,\ \text{tails} = 0\}$

3 successive trials:
P[Heads] = P[Tails] = p = 0.5 at every stage.

$(0.5)^3 = 0.125 = \frac{1}{8}$

(EACH OF THE 8 OUTCOMES HAS AN EQUAL PROBABILITY OF OCCURRING)



HHH $X = 3$ $P[X=3] = \frac{1}{8}$

HHT $X = 2$

HTH $X = 2$ $P[X=2] = \frac{1}{8}\cdot 3 = \frac{3}{8}$

HTT $X = 1$

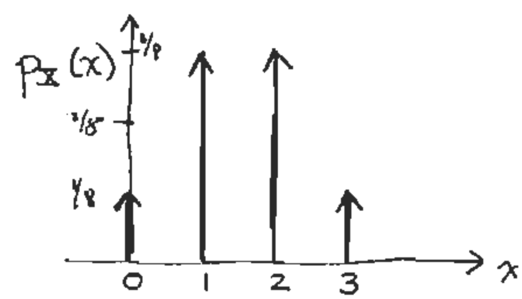
THH $X = 2$ $P[X=1] = \frac{1}{8}\cdot 3 = \frac{3}{8}$

THT $X = 1$

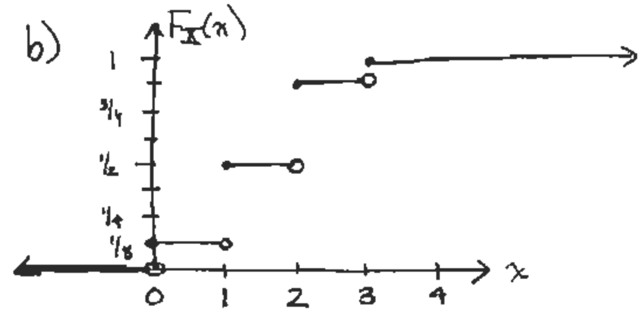
TTH $X = 1$

TTT $X = 0$ $P[X=0] = \frac{1}{8}$

a)



b)



* Note there is a different scale on the $p_X(x)$ and $F_X(x)$ plots. *

c) $$\mu_x = \sum_{n=0}^{3} x \cdot p_X(x) = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{12}{8} = \boxed{1.5 = \mu_x}$$

$$\sigma_x^2 = \sum_{n=0}^{3}(x^2 - \mu_x^2)p_X(x) = \sum_{n=0}^{3} x^2 p_X(x) - \mu_x^2 = 0^2\cdot\frac{1}{8} + 1^2\cdot\frac{3}{8} + 2^2\cdot\frac{3}{8} + 3^2\cdot\frac{1}{8} - (1.5)^2$$

$$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} - \frac{9}{4} = \frac{24}{8} - \frac{9}{4} = \frac{12}{4} - \frac{9}{4} = \frac{3}{4} \qquad \boxed{0.75 = \sigma_x^2}$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = \boxed{0.866 = \sigma_x}$$

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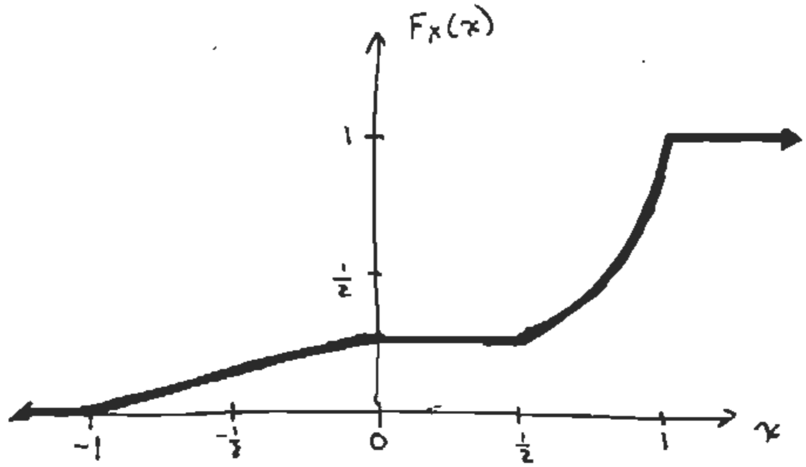
4)



a) $P(X \le \frac{1}{2}) = F_X(\frac{1}{2}) = \boxed{\frac{1}{4}}$

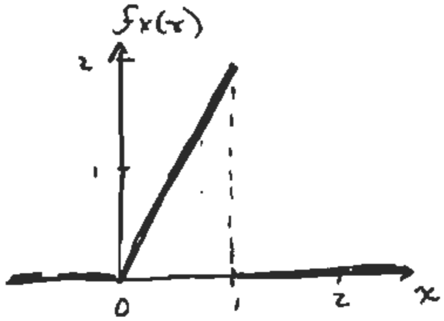
b) $P(X \ge 0) = 1 - F_X(0) = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$

c) $P(0 \le X < \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(0) = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

d) $P(X \le \frac{3}{4}) = F_X(\frac{3}{4}) = (\frac{3}{4})^2 = \boxed{\frac{9}{16}}$

e) $P(X \ge 1) = 1 - F_X(1) = 1 - 1 = \boxed{0}$

5)



a) $\mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$= \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^1 x \cdot 2x \cdot dx + \int_1^{\infty} x \cdot 0 \cdot dx$

$= \int_0^1 2x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}(1-0) = \boxed{\frac{2}{3} = \mu_X}$

b) $\sigma_X \Rightarrow \sigma_X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - (\mu_X)^2 = \int_0^1 x^2 \cdot 2x \cdot dx - (\frac{2}{3})^2 = \int_0^1 2x^3 dx - \frac{4}{9}$

[USE $\sigma_X^2 = E[x^2] - \mu_X^2$]

$= 2\frac{x^4}{4}\Big|_0^1 - \frac{4}{9} = \frac{2}{4}(1-0) - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \frac{1}{18} = \sigma_X^2$

$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = \boxed{\frac{\sqrt{2}}{6} = \sigma_X}$

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6) $$\eta(t) = A\cos(\omega t + \phi)$$

a) A is Gaussian with zero mean ∴

$$f_A(a) = \frac{1}{\sigma_A\sqrt{2\pi}} e^{\frac{-a^2}{2\sigma_A^2}}$$

b) $p_d = \rho g A e^{kz}\cos(\omega t + \phi)$ at $z=0$ $p_d = \rho g A\cos(\omega t + \phi)$

c) To determine if $p_d(t)$ is a stationary process, you must check if the mean and variation and correlation are independent of time.

$$\mu_{pd}(t) = E[p_d(t)] = E[\rho g A\cos(\omega t + \phi)]$$

$$= E[A]\cdot \rho g \cos(\omega t + \phi) = 0 \quad \therefore \mu_{pd} = 0 = \text{constant}$$

but I know that A has zero mean, i.e. $E[A]=0$

$$\sigma_{pd}^2(t) = E[(p_d(t) - \mu_{pd})^2] = E[(p_d(t))^2] = E[(\rho g)^2 A^2 \cos^2(\omega t + \phi)]$$

$$= (\rho g)^2\cos^2(\omega t + \phi)\, E[A^2] = (\rho g)^2\cos^2(\omega t + \phi)\int_{-\infty}^{\infty} a^2 f_A(a)\,da$$

$$= (\rho g)^2\cos^2(\omega t + \phi)\,\sigma_a^2 \quad \therefore \sigma_{pd}^2 \text{ IS A FUNCTION OF TIME}$$

∴ I know that it is <u>NOT</u> a stationary process

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7) $$\eta(t) = \sum_{i=1}^{N} A_i \cos(\omega_i t + \phi_i)$$

FROM THE SPECTRUM PLOT AND THE EQ'N: $S_\eta(\omega)\,\delta\omega = \frac{1}{2}A_i^2$; $\delta\omega = 1\ \frac{rad}{sec}$

For $\omega_1 = 1\ \frac{rad}{sec}$, $\frac{1}{2}A_1^2 = (12)(1)$ so $A_1^2 = 24$, $A_1 = 2\sqrt{6}$

For $\omega_2 = 2\ \frac{rad}{sec}$, $\frac{1}{2}A_2^2 = (18)(1)$ so $A_2^2 = 36$, $A_2 = 6$

For $\omega_3 = 3\ \frac{rad}{sec}$, $\frac{1}{2}A_3^2 = (14)(1)$ so $A_3^2 = 28$, $A_3 = 2\sqrt{7}$

For $\omega_4 = 4\ \frac{rad}{sec}$, $\frac{1}{2}A_4^2 = (8)(1)$ so $A_4^2 = 16$, $A_4 = 4$

Therefore:
$$\eta(t) = 2\sqrt{6}\cos(t+\phi_1) + 6\cos(2t+\phi_2) + 2\sqrt{7}\cos(3t+\phi_3) + 4\cos(4t+\phi_4)$$

USE RAND FUNCTION IN MATLAB TO GENERATE ϕ VALUES. IT IS OKAY TO EITHER USE THE RAND FUNCTION TO CREATE 40 VALUES OF ϕ: 4 ϕ's IN EACH OF 10 EQUATIONS OR TO GENERATE 10 VALUES OF ϕ AND USE EACH ONE 4 TIMES IN EACH $\eta(t)$ EQUATION.

SEE ATTACHED MATLAB PLOTS.

8) a) The majority of sea waves are caused by wind.

b) The phase speed of waves is maximized when it matches wind speed. Through the deep water dispersion relation, wavelength is limited for any frequency.

c) Significant wave height is the average of the 1/3 highest waves.

d) Significant wave height correlates very closely to observations of wave height by experienced mariners. Therefore, historical observation data is usually significant wave height.

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Homework 3 Problem 7: Sample MATLAB Code

```
t = 0:90;
w = [1 2 3 4];
A = [sqrt(24) sqrt(36) sqrt(28) sqrt(16)];
for j=1:10
for i=1:4
    phi = 2*pi*rand(1);
    %This randomly generates a separate phi value for each of the four
    %waves.
    WAVE(i,:) = A(i)*cos(w(i)*t+phi);
    %This wave equation iterates four times and generates waves for each of
    %the frequencies and their corresponding amplitudes.
end
S(j,:) = WAVE(1,:)+WAVE(2,:)+WAVE(3,:)+WAVE(4,:);
%This statement generates my wave elevation as a sum of the four waves
%generated above in each iteration.  This loop iterates 10 times, created 10
separate
%realizations of the wave elevation equation with random phase shifts.
%
%
Mean_Ensemble = MEAN(S(:,30));
Variance_Ensemble = VAR(S(:,30));
Mean_Temporal(j,:) = MEAN(S(j,:));
Variance_Temporal(j,:) = VAR(S(j,:));
end
subplot(5,2,1), plot(t,S(1,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,2), plot(t,S(2,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,3), plot(t,S(3,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,4), plot(t,S(4,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,5), plot(t,S(5,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,6), plot(t,S(6,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,7), plot(t,S(7,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,8), plot(t,S(8,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,9), plot(t,S(9,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
subplot(5,2,10), plot(t,S(10,:))
XLABEL('Time (s)')
YLABEL('Wave Elevation (m)')
Mean_Ensemble
Variance_Ensemble
Mean_Temporal
Variance_Temporal
```

Example Output from Above Code (Note: It changes every run due to random realizations of phi)

```
HW37

Mean_Ensemble =

    1.7494


Variance_Ensemble =

   54.5025


Mean_Temporal =

    0.0968
   -0.0551
    0.1645
    0.0659
    0.0887
   -0.1158
   -0.0079
   -0.0449
   -0.1013
   -0.1384


Variance_Temporal =

   52.7553
   54.5923
   51.9517
   53.4987
   52.7898
   53.2102
   51.6055
   51.3866
   52.4914
   52.8724

diary off
```

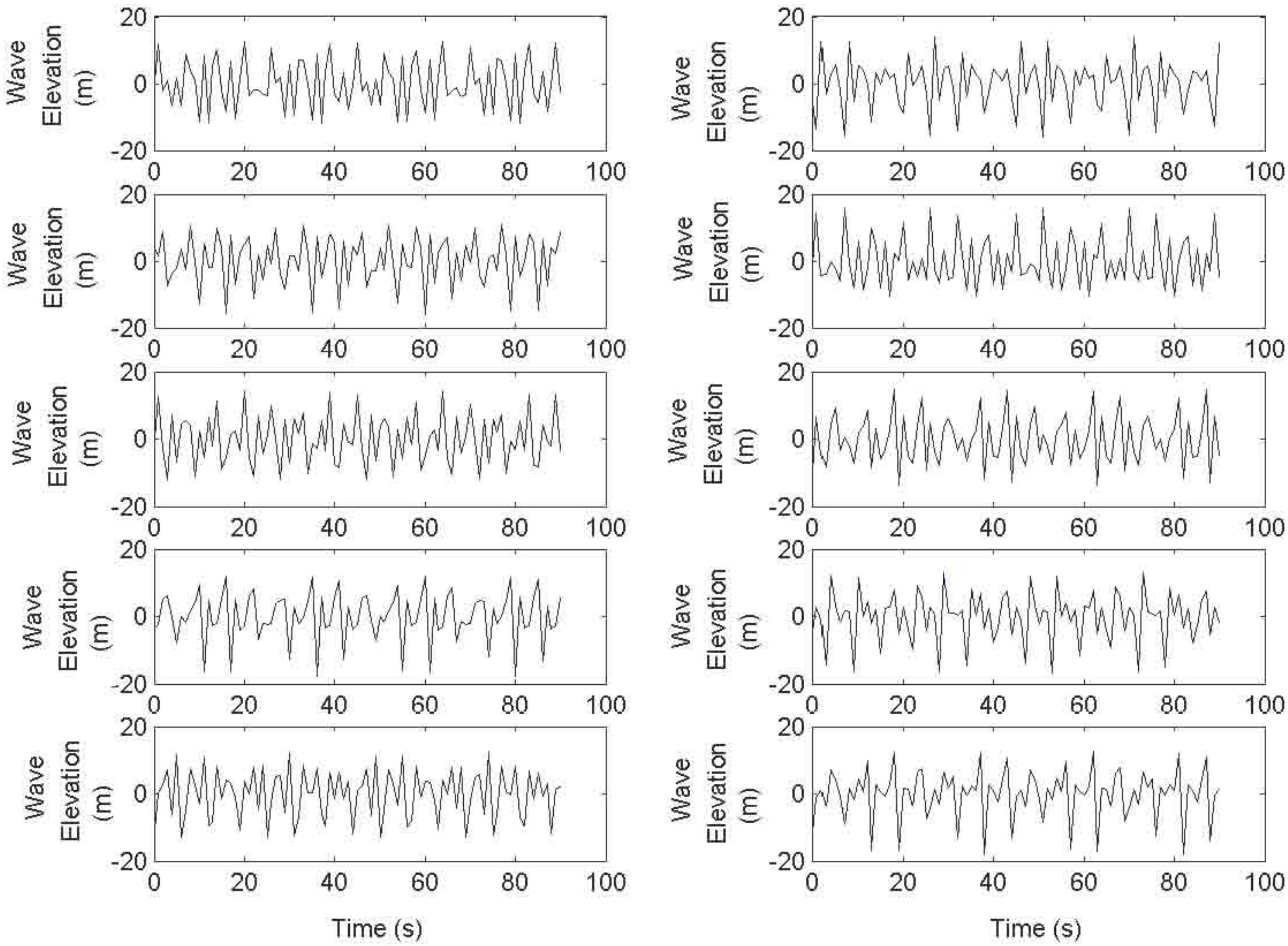
Comments:

The ensemble statistics are computed over 10 data points, S(:,30), which are each of the wave elevation realizations at time t = 30 seconds.

The temporal statistics are computed 10 times, for 10 separate wave elevation realizations, over 91 data points which represent a time range of t = [0:90].

Therefore, the temporal statistics are more consistent from one execution of the m-file to the next.

Wave elevation, with a uniformly distributed phi over a 2π interval, is a stationary, ergodic random process. Therefore the ensemble statistics should equal the temporal statistics. Over multiple m-file executions, there will be more variability in the ensemble statistics, but they should approximate the temporal statistics over multiple iterations.



Wave Elevation (m)
Time (s)