13.42 HW#4 SOL'NS

1. BAG 1 - 3 G, 4 R

   BAG 2 - 5 G, 1 R

LET $P(G) = P(\text{CHOSE GREEN MARBLE})$

$P(R) = P(\text{" RED "})$

$P(A_1) = P(\text{CHOSE BAG 1})$

$P(A_2) = P(\text{" " 2})$

$$P(A_1 \mid G) = \frac{P(G \mid A_1)P(A_1)}{P(G \mid A_1)P(A_1) + P(G \mid A_2)P(A_2)} \quad \text{(BAYES' TH'M)}$$

$$= \frac{(3/7)(1/2)}{(3/7)(1/2) + (5/6)(1/2)}$$

$$= \frac{18}{53}.$$

2. GIVEN $\xi(t) = A\cos(\omega t + \phi)$, DETERMINE IF STATIONARY:

- $\mu_\xi(t) = E\{\xi(t)\}$

$$= E\{A\cos(\omega t + \phi)\}$$

$$= \cos(\omega t + \phi)\, E\{A\}$$

$$= \cos(\omega t + \phi) \int_{-\infty}^{\infty} a \overbrace{p_A(a)}^{\text{GAUSSIAN}}\, da$$

$$= 0 \longrightarrow \text{CONST.} \checkmark$$

- $\sigma_\xi^2(t) = E\{[\xi(t) - \cancelto{0}{\mu_\xi}]^2\}$

$$= E\{A^2\cos^2(\omega t + \phi)\}$$

$$= \cos^2(\omega t + \phi) \int_{-\infty}^{\infty} a^2 p_A(a)\, da$$

$$= \cos^2(\omega t + \phi) \cdot \sigma^2 \longrightarrow \text{NOT INDEP. OF } t$$

$\therefore$ $\underline{\xi(t) \text{ IS NOT STATIONARY.}}$

3a. $\underline{\mu_Y(t)} = E\{Y(t)\}$

$$= E\{h(t) * U(t)\}$$

$$= E\left\{\int_{-\infty}^{\infty} h(\tau)\, U(t-\tau)\, d\tau\right\}$$

$$\left[\quad E\{g(U)\} = \int_{-\infty}^{\infty} g(u)\, p_U(u)\, du \right.$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)\, U(t-\tau)\, d\tau\right] p_U(u)\, du$$

$$= \int_{-\infty}^{\infty} h(\tau) \underbrace{\int_{-\infty}^{\infty} U(t-\tau)\, p_U(u)\, du}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E\{U(t-\tau)\}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mu_U(t-\tau)\, d\tau$$

$$\underline{= h(t) * \mu_U(t)}$$

b. $h(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$, $\mu_U(t) = \mu_0$

FIND $\mu_Y(t)$.

FROM 3a,

$$\mu_Y(t) = h(t) * \mu_0$$

$$= \int_{-\infty}^{\infty} h(\tau)\,\mu_0\,d\tau$$

$$= \mu_0\,H(0)$$

$$\left[\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)\,e^{-i\omega t}\,dt \\ &= \int_{0}^{\infty} e^{-(a+i\omega)t}\,dt \\ &= \frac{1}{a+i\omega} \end{aligned}\right.$$

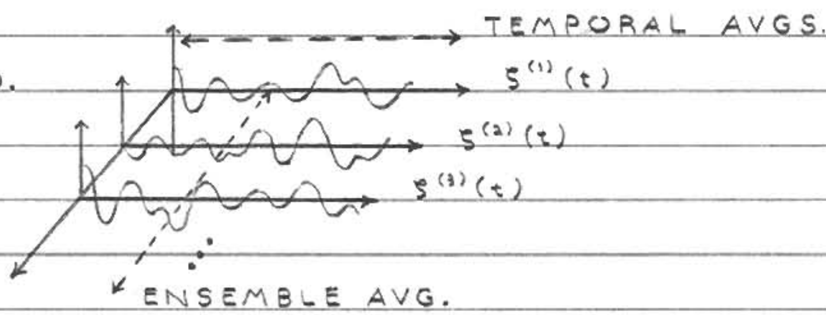
$$= \underline{\frac{\mu_0}{a}}.$$

4. $S_\xi(\omega_i)\,\delta\omega = \frac{1}{2} A_i^2$

$$A_i = \sqrt{2\,S_\xi(\omega_i)\,\delta\omega}$$

- $\omega_1 = 0.5\ \text{rad/s}$; $A_1 = 3.16\ m$
  $\omega_2 = 1.0$ " ; $A_2 = 4.47\ m$
  $\omega_3 = 1.5$ " ; $A_3 = 3.87\ m$
  $\omega_4 = 2.0$ " ; $A_4 = 2.24\ m$

a & b.



5a. $E\{[X(t+\tau)-X(t)]^2\} =$

$$= E\{X^2(t+\tau) - 2X(t)X(t+\tau) + X^2(t)\}$$

$$= E\{X^2(t+\tau)\} + E\{X^2(t)\} - 2E\{X(t)X(t+\tau)\}$$

$$= R_{xx}(0) + R_{xx}(0) - 2R_{xx}(\tau)$$

$$= 2[R_{xx}(0) - R_{xx}(\tau)].$$

b. WIENER-KHINCHINE RELATION:

$$S_{YY}(\omega) = |H(\omega)|^2 S_{xx}(\omega) \qquad (1)$$

- FIND F.T. OF $h(t)$:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\,dt = \int_0^{\infty} e^{-(a+i\omega)t}\,dt = \frac{1}{a+i\omega}$$

- FIND F.T. OF $R_{xx}(\tau)$:

$$S_{xx}(\omega) = \tilde{R}_{xx}(\tau) = \int_{-\infty}^{\infty} Ce^{-k|\tau|}e^{-i\omega\tau}\,d\tau$$

$$\left[\; R_{xx}(\tau) = \begin{cases} e^{-k\tau}, & \tau > 0 \\ e^{k\tau}, & \tau < 0 \end{cases}\right.$$

$$= C\left(\int_0^{\infty} e^{-k\tau}e^{-i\omega\tau}\,d\tau + \int_{-\infty}^{0} e^{k\tau}e^{-i\omega\tau}\,d\tau\right)$$

$$= C\left(\frac{1}{k+i\omega} + \frac{1}{k-i\omega}\right)$$

$$= C \left[\frac{1}{k+i\omega} \frac{k-i\omega}{k-i\omega} + \frac{1}{k-i\omega} \frac{k+i\omega}{k+i\omega} \right]$$

$$= C \frac{2k}{k^2+\omega^2} .$$

- INVOKING EQ. (1),

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \left(\frac{1}{a+i\omega}\right)\left(\frac{1}{a+i\omega}\right)^* C \frac{2k}{k^2+\omega^2}$$

$$= \left(\frac{1}{a^2+\omega^2}\right) C \frac{2k}{k^2+\omega^2} \qquad (2)$$

- BY PARTIAL FRACTIONS THE R.H.S. MAY BE REWRITTEN:

$$\frac{2Ck}{(a^2+\omega^2)(k^2+\omega^2)} = \frac{A}{a^2+\omega^2} + \frac{B}{k^2+\omega^2}$$

$$2Ck = A(k^2+\omega^2) + B(a^2+\omega^2)$$

- SET $\omega = ik$: $\quad B = \dfrac{2Ck}{a^2-k^2}$

- SET $\omega = ia$: $\quad A = \dfrac{2Ck}{k^2-a^2}$

THEN EQ. (2) BECOMES

$$S_{YY}(\omega) = \underset{\times \frac{a}{a}}{\frac{2Ck}{(k^2-a^2)(a^2+\omega^2)}} + \frac{2Ck}{(a^2-k^2)(k^2+\omega^2)}$$

$$= \underbrace{\frac{2a}{a^2+\omega^2}}_{①} \left(\frac{Ck}{k^2-a^2} \cdot \frac{1}{a} \right) + \underbrace{\frac{2k}{k^2+\omega^2}}_{②} \left(\frac{C}{a^2-k^2} \right)$$

- NOW NOTICE THAT TERMS ① & ② ARE OF

THE SAME FORM AS $\tilde{R}_{xx}(\tau)$!

$\therefore$ $R_{YY}(\tau) = \text{I.F.T.}\left\{ S_{YY}(\omega) \right\}$

$$= \frac{Ck}{a(k^2 - a^2)} e^{-a|\tau|} + \frac{C}{a^2 - k^2} e^{-k|\tau|} .$$

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