Problem 4.25
This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin
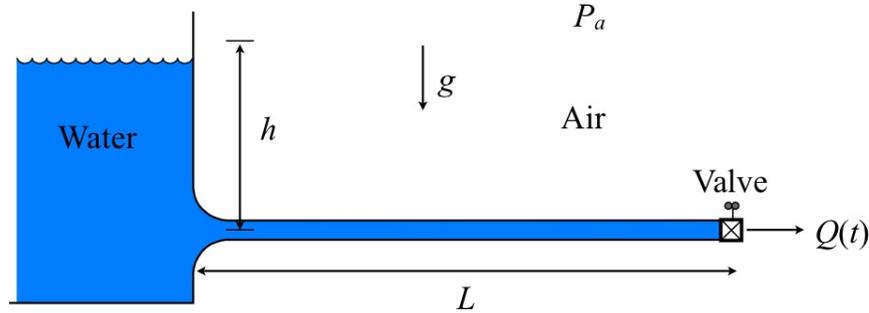
Water flows from a large reservoir through a very long pipe under constant head h. When the valve is slowly closed, the head h remains constant, but the volume flow rate is reduced.

a) Neglecting friction and compressibility of the water, demonstrate that the gage pressure just upstream of the valve at any instant during the closure period is given approximately by

$$p = \rho \left(gh - \frac{Q^2}{2A^2} - \frac{L}{A}\frac{dQ}{dt} \right) \tag{4.25a}$$

where A is the cross-sectional area of the pipe.

b) Suppose the "valve" is a short, frictionless nozzle with variable exit area $A_e(t)$. At $t < 0$, prior to valve actuation, a steady flow takes place with $A_e = A$. It is desired to program the valve closure such that the volume flow rate decreases linearly in time from its initial steady-state value to zero in a period of τ. Show that this requires that $A_e(t)$ be programmed such that

$$\frac{A_e(t)}{A} = \left(1 - \frac{t}{\tau} \right) \left[1 + \left(\frac{L}{\tau} \right) \left(\frac{2}{gh} \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} \tag{4.25b}$$

Solution:
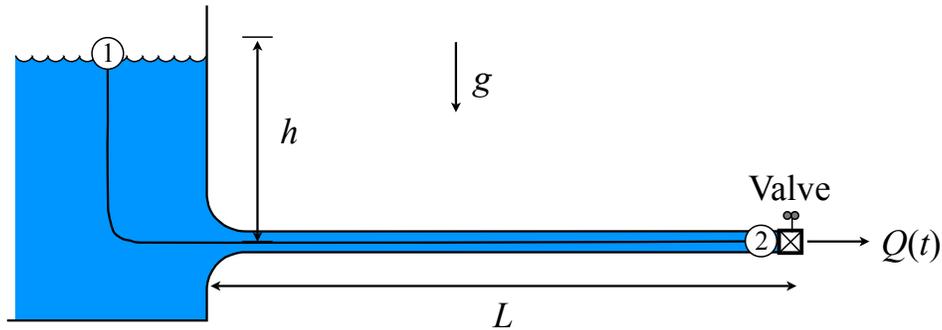
a) If we assume that the flow is inviscid, irrotational and incompressible, but not steady, we may apply the unsteady Bernoulli equation along a streamline between stations 1 and 2, which takes the form

$$\rho \int_{s1}^{s2} \frac{\partial v}{\partial t} ds + \left(p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2 \right) - \left(p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 \right) = 0 \qquad (4.25\text{c})$$

We define the gage pressure $p = p_2 - p_1$. Additionally, we recognize that fluid speed at station 1 is essentially zero because h remains constant in time since the reservoir area is much greater than the pipe area A. Finally we note that the speed of the fluid in the horizontal pipe is spatially uniform along its length, L, but changes in time, so we obtain

$$\rho \frac{\partial v}{\partial t} L + p + \frac{1}{2}\rho v_2^2 - \rho g h = 0 \qquad (4.25\text{d})$$

The speed at station 2 is related to the volume flow rate by $v_2 = Q/A$, so when we substitute in this result into Eq. (4.25d) and rearrange this result, we find

$$\boxed{p = \rho \left(gh - \frac{Q^2}{2A^2} - \frac{L}{A}\frac{dQ}{dt} \right)} \qquad (4.25\text{e})$$

b) Consider now a third station, station 3, which is just downstream of the valve where the pressure is once again atmospheric, such that $p_3 = p_1$. Our objective is to have a volumetric flow rate $Q(t)$ that decreases linearly in time to zero in a period τ, therefore

$$Q(t) = Q_0 \left(1 - \frac{t}{\tau} \right) \tag{4.25f}$$

where Q_0 is the initial steady state flow rate before the valve begins to close. This flow rate, Q_0, is determined by applying the steady Bernoulli equation from station 1 to station 3, and it is

$$Q_0 = \sqrt{2gh}A \tag{4.25g}$$

following the same assumptions of negligible velocity at station 1, as before.

To proceed, we assume that the flow between stations 2 and 3 is pseudo-steady and apply the steady Bernoulli equation between each point (a justification for this assumption will be offered later):

$$p_2 - p_3 = \frac{1}{2}\rho v_3^2 - \frac{1}{2}\rho v_2^2 \tag{4.25h}$$

Since $p_2 - p_3$ is equal to the gage pressure, we may replace it with Eq. (4.25e) and rewrite speeds in terms of volumetric flow rates to obtain

$$\rho \left(gh - \frac{Q^2}{2A^2} - \frac{L}{A}\frac{dQ}{dt} \right) = \frac{1}{2}\rho \left(\frac{Q^2}{A_e^2} - \frac{Q^2}{A^2} \right) \tag{4.25i}$$

which simplifies to

$$gh - \frac{L}{A}\frac{dQ}{dt} = \frac{1}{2}\frac{Q^2}{A_e^2} \tag{4.25j}$$

Now we may substitute for Q in Eq. (4.25i) with (4.25f) and its derivative and (4.25g) to obtain

$$gh - \frac{L}{A}\left(-\frac{1}{\tau}\sqrt{2gh}A \right) = \frac{1}{2}\frac{2ghA^2 \left(1 - \frac{t}{\tau} \right)^2}{A_e^2} \tag{4.25k}$$

This expression may be rearranged to obtain our final result in the desired form

$$\boxed{\frac{A_e(t)}{A} = \left(1 - \frac{t}{\tau} \right) \left[1 + \left(\frac{L}{\tau} \right) \left(\frac{2}{gh} \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}}} \tag{4.25l}$$

Returning now to our earlier assumption that the flow between stations 2 and 3 was steady, we consider the relative magnitude of the unsteady term in the unsteady Bernoulli equation based on our results from the pseudo-steady assumption, were we to have included it in Eq. (4.25h). The unsteady Bernoulli equation between stations 2 and 3 is:

$$p_2 - p_3 = \rho \int_{s2}^{s3} \frac{\partial v}{\partial t} ds + \frac{1}{2}\rho v_3^2 - \frac{1}{2}\rho v_2^2 \tag{4.25m}$$

We must compare the relative magnitudes of the unsteady term with the difference between the square of the velocities. If we assume that the characeristic velocity in the unsteady term is $v_3 = Q(t)/A_e(t)$, then

$$\rho \int_{s2}^{s3} \frac{\partial v}{\partial t} ds = \rho \frac{d}{dt}\left(\frac{Q(t)}{A_e(t)}\right) l \tag{4.25n}$$

where l is the length of the nozzle. Since $Q(t) \sim (1 - t/\tau)$ and $A_e(t) \sim (1 - t/\tau)$, $\frac{d}{dt}\left(\frac{Q(t)}{A_e(t)}\right) = 0$ for all t and its exclusion from Eq. (4.25h) is clearly justified. If, on the other hand, we were to have taken v_2 as the characteristic velocity in the integral, then

$$\rho \int_{s2}^{s3} \frac{\partial v}{\partial t} ds = \rho \frac{d}{dt}\left(\frac{Q(t)}{A}\right) l = -\rho\left(\frac{Q_0}{A\tau}\right) l = -\rho\left(\frac{\sqrt{2gh}}{\tau}\right) l \tag{4.25o}$$

The difference between the square of the velocities is

$$\frac{1}{2}\rho(v_3^2 - v_2^2) = \frac{1}{2}\rho\left[\left(\frac{Q_0(1-\frac{t}{\tau})}{A(1-\frac{t}{\tau})[1+\frac{L}{\tau}(\frac{2}{gh})^{\frac{1}{2}}]^{-\frac{1}{2}}}\right)^2 - \left(\frac{Q_0(1-\frac{t}{\tau})}{A}\right)^2\right] \tag{4.25p}$$

This result simplifies to

$$\frac{1}{2}\rho(v_3^2 - v_2^2) = \rho gh\left[1 + \frac{L}{\tau}\left(\frac{2}{gh}\right)^{\frac{1}{2}} - \left(1 - \frac{t}{\tau}\right)^2\right] \tag{4.25q}$$

This term will be smallest at $t = 0$, when it equals

$$\frac{1}{2}\rho(v_3^2 - v_2^2)_{t=0} = \rho\left(\frac{\sqrt{2gh}}{\tau}\right) L \tag{4.25r}$$

Comparing the magnitudes of Eq. (4.25o) and (4.25r), we conclude that provided $L \gg l$, the exclusion of the unsteady term from Eq. (4.25h) is justified.

$\square$

Problem Solution by TJO, Fall 2010

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