MIT Department of Mechanical Engineering

# 2.25 Advanced Fluid Mechanics

## Problem 7.14

*This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin*

A fluid of density $\rho$ and viscosity $\mu$ flows through a long pipe of diameter $D$ at the volume flow rate $Q$.

(a) Demonstrate that if the flow is laminar (*i.e.*, totally steady) and fully developed (the velocity profile and the pressure gradient no longer change with downstream distance $x$), the pressure gradient in the direction of flow must have the form

$$\frac{dp}{dx} = -K\frac{Q\mu}{D^4}$$

where $K$ is constant.

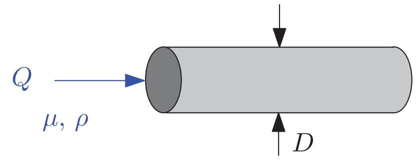
(b) Demonstrate that if the flow is turbulent (*i.e.*, unsteady) but steady and fully developed in the mean, the mean pressure gradient in the direction of flow must have the form

$$\frac{dp}{dx} = \left(\frac{\rho Q^2}{D^5}\right) \cdot \phi\left(\frac{\rho Q}{\mu D}\right)$$

## Solution:

Consider flow through a pipe:




| $\frac{dp}{dx}$ | $Q$ | $D$ | $\rho$ | $\mu$ |
|---|---|---|---|---|
| $[\mathrm{ML^{-2}T^{-2}}]$ | $[\mathrm{L^3T^{-1}}]$ | $[\mathrm{L}]$ | $[\mathrm{ML^{-3}}]$ | $[\mathrm{ML^{-1}T^{-1}}]$ |

(a) Laminar Case

Steady and fully-developed $\Rightarrow$ INERTIA-FREE and $dp/dx \neq f(\rho)$

$$\therefore \frac{dp}{dx} = f_1(\mu, Q, D)$$
$$n = 4 \text{ variables}$$
$$k = 3 \text{ primary variables}$$
$$\Rightarrow j = 1 \text{ dimensionless group}$$

Our primary variables must be $\mu$, $Q$, and $D$ (since there are no other variables)

$$\Pi = \frac{dp}{dx}\mu^a Q^b D^c$$
$$[\,] = [\mathrm{M^0L^0T^0}] = [\mathrm{ML^{-2}T^{-2}}][\mathrm{ML^{-1}T^{-1}}]^a[\mathrm{L^3T^{-1}}]^b[\mathrm{L}]^c$$
$$\mathrm{M^0 = M^1M^a} \qquad \Rightarrow a = -1$$
$$\mathrm{T^0 = T^{-2}T^1T^{-b}} \qquad \Rightarrow b = -1$$
$$\mathrm{L^0 = L^{-2}L^1L^{-3}L^c} \qquad \Rightarrow c = 4$$
$$\Rightarrow \Pi = \frac{dp}{dx} \cdot \frac{D^4}{\mu Q} = \text{constant}$$

Keep in mind that $\frac{dp}{dx}$ and $Q$ have opposite signs (*i.e.*, $\frac{dp}{dx} > 0$ when $Q < 0$ and vice versa). Therefore, define constant $-K$ to ensure that $K \geq 0$.

$$\Rightarrow \boxed{\frac{dp}{dx} = -K\frac{\mu Q}{D^4}}$$

(b) Turbulent Case

Steady, and fully-developed in the mean

$$\therefore \frac{dp}{dx} = f_2(\mu, \rho, Q, D)$$
$$n = 5 \text{ variables}$$
$$k = 3 \text{ primary variables}$$
$$\Rightarrow j = 2 \text{ dimensionless groups}$$

For our primary variables, we choose (1) a fluid property: $\rho$, (2) a flow parameter: $Q$, and (3) a geometric parameter: $D$. Follow the same procedure as in part (a) to find

$$\Pi_1 = \frac{dp}{dx} \cdot \frac{D^5}{\rho Q^2}$$
$$\Pi_2 = \frac{\mu D}{\rho Q}$$

Since $\Pi_1 = \phi(\Pi_2)$,

$$\boxed{\frac{dp}{dx} = \frac{\rho Q^2}{D^5}\,\phi\left(\frac{\mu D}{\rho Q}\right)}$$

$\square$

*Problem Solution by Sungyon Lee, Fall 2005*