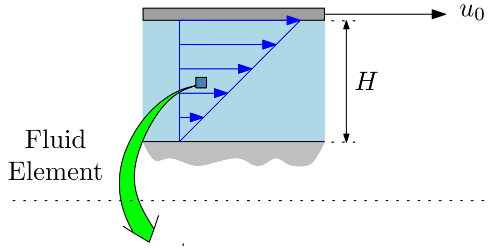
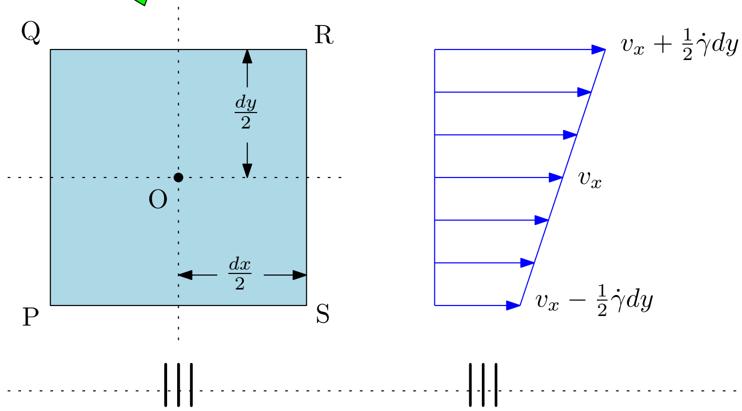
Geometric Interpretation of Fluid Kinematics
In Steady Shear Flow



$$\dot\gamma = \frac{u_0}{H} \qquad \dot\gamma = \frac{\partial v_x}{\partial y}$$

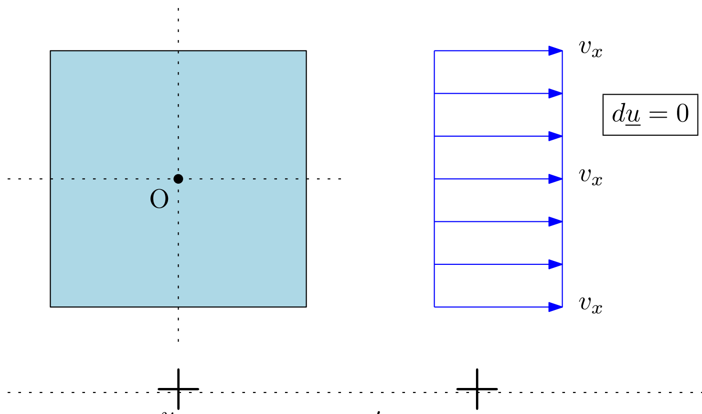
$$\begin{cases} v_x = \dot\gamma y \\ v_y = 0 \\ v_z = 0 \end{cases}$$



Steady Shear Flow

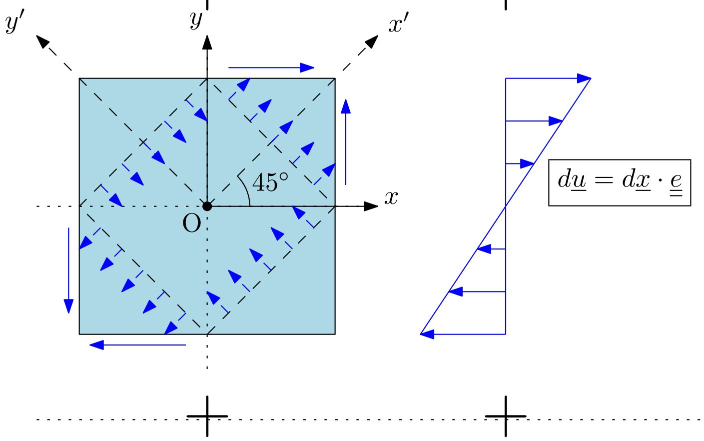
$$\underline{\nabla u} = \begin{pmatrix} 0 & 0 & 0 \\ \dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{u}(\underline{x}+d\underline{x}) = \underline{u}(\underline{x}) + d\underline{x}\cdot(\underline{\underline{e}}+\underline{\underline{\Omega}})$$



Steady Translation

$$\underline{u}(\underline{x}+d\underline{x}) = \underline{u}(\underline{x})$$

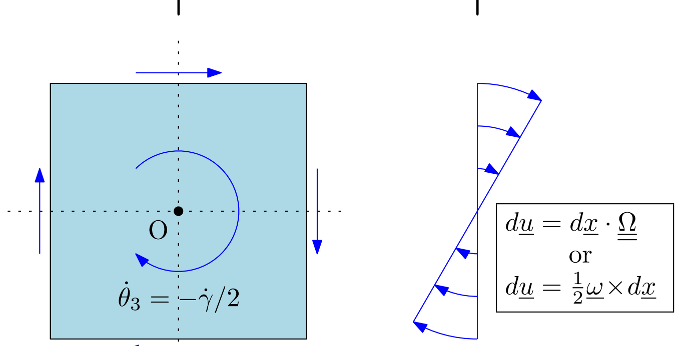


Pure Deformation

$$\underline{\underline{e}} = \begin{pmatrix} 0 & \dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rotation through 45° ($\underline{\underline{e}}' = \underline{\underline{C}}\,\underline{\underline{e}}$)
⇒ Principal Axes:

$$\underline{\underline{C}} = \begin{pmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{e}}' = \begin{pmatrix} \frac{\dot\gamma}{2} & 0 & 0 \\ 0 & -\frac{\dot\gamma}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



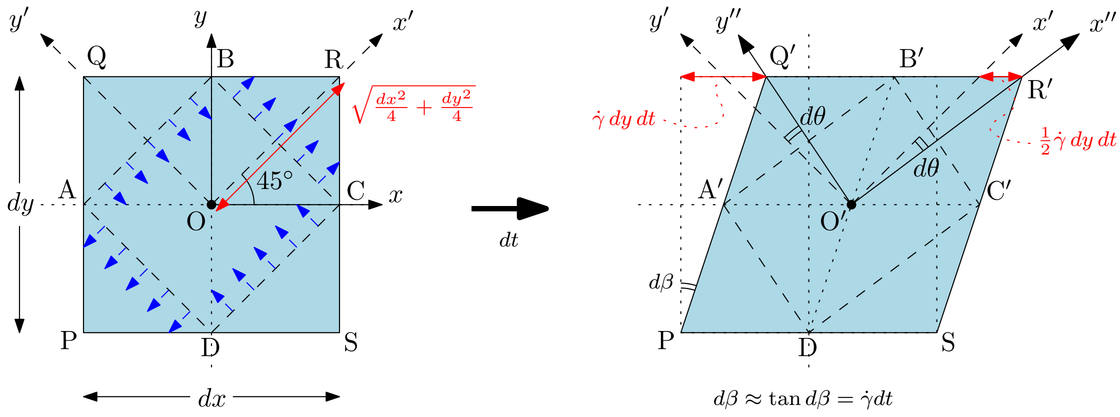
Pure Rotation

$$\underline{\underline{\Omega}} = \begin{pmatrix} 0 & -\dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot\gamma \end{pmatrix} = -\epsilon_{ijk}\Omega_{jk}$$

(Vorticity) = 2×(Angular Velocity)

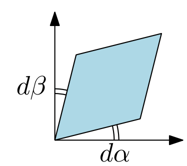
Consider the deformation in a (small) time dt:



<u>In x-y coordinate frame:</u> deformation is *simple shear*:

$$\underline{\underline{e}} = \begin{pmatrix} 0 & \dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Displacement: $(Q \to Q') = v_Q dt = \dot\gamma\, dy\, dt$
- Length: $PQ' = PQ\sqrt{1+(\dot\gamma dt)^2} = dy\sqrt{1+(\dot\gamma dt)^2}$
- Average Angular Velocity: $\dot\theta_3 = \frac{1}{2}\left[\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right] = \frac{1}{2}\left[0 - \frac{\dot\gamma dt}{dt}\right] = -\frac{\dot\gamma}{2}$



x

<u>In x'-y' coordinate frame:</u> deformation is *extensional*:

$$\underline{\underline{e}}' = \begin{pmatrix} \dot\gamma/2 & & 0 \\ 0 & -\dot\gamma/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \frac{\partial v_{x'}}{\partial x'} = \frac{\dot\gamma}{2}, \qquad \frac{\partial v_{y'}}{\partial y'} = -\frac{\dot\gamma}{2}$$

Line Segment:

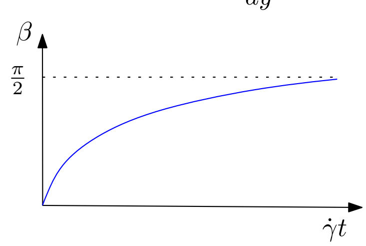
$$A'B' = AB + 2\left(\frac{\partial v_{x'}}{\partial x'}\right) dx'dt = AB + \dot\gamma dx'dt$$

$$B'C' = BC + 2\left(\frac{\partial v_{y'}}{\partial y'}\right) dy'dt = BC - \dot\gamma dy'dt$$

In addition, axes rotates by $d\theta = -\frac{1}{2}\dot\gamma dt \quad \Rightarrow \quad \dot\theta = -\frac{1}{2}\dot\gamma$ from $x'y' \to x''y''$

<u>Note</u> that expressions for angular displacement are only valid for <u>small</u> dt such that $\tan d\beta \approx d\beta$
$\Rightarrow$ In the limit of finite time, the change in the (initially) perpendicular line segments QPS is:

$$d\tan\beta = \frac{\dot\gamma\, dy\, dt}{dy} \quad \Rightarrow \quad \boxed{\beta = \tan^{-1}(\dot\gamma t)}$$



2.25 Advanced Fluid Mechanics
Fall 2013