2.29 Numerical Fluid Mechanics
Spring 2015 – **Lecture 19**

REVIEW Lecture 18:

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla.(\rho \vec{v}\ \vec{v}) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

$$\nabla.\vec{v} = 0$$

- Solution of the Navier-Stokes Equations
  - Discretization of the convective and viscous terms
  - Discretization of the pressure term $\quad \tilde{p} = p - \rho\, \mathbf{g}.\mathbf{r} + \mu \frac{2}{3} \nabla.\mathbf{u} \quad (p\, \vec{e}_i - \rho g_i x_i \vec{e}_i + \frac{2}{3} \mu \frac{\partial u_j}{\partial x_j} \vec{e}_i)$

    $$\int_S -\tilde{p}\ \vec{e}_i.\vec{n} dS$$
  - Conservation principles
    - Momentum and Mass
    - Energy

$$\frac{\partial}{\partial t} \int_{CV} \rho \frac{\|\vec{v}\|^2}{2} dV = -\int_{CS} \rho \frac{\|\vec{v}\|^2}{2} (\vec{v}.\vec{n}) dA - \int_{CS} p\ \vec{v}.\vec{n}\, dA + \int_{CS} (\vec{\vec{\varepsilon}}.\vec{v}).\vec{n}\, dA + \int_{CV} \left(-\vec{\vec{\varepsilon}} : \nabla \vec{v} + p \nabla.\vec{v} + \rho \vec{g}.\vec{v} \right) dV$$

  - Choice of Variable Arrangement on the Grid
    - Collocated and Staggered
  - Calculation of the Pressure

$$\nabla.\nabla p = \nabla^2 p = -\nabla.\frac{\partial \rho \vec{v}}{\partial t} - \nabla.\left(\nabla.(\rho \vec{v}\ \vec{v})\right) + \nabla.\left(\mu \nabla^2 \vec{v}\right) + \nabla.\left(\rho \vec{g}\right) = -\nabla.\left(\nabla.(\rho \vec{v}\ \vec{v})\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}\left(\frac{\partial \left(\rho u_i u_j\right)}{\partial x_j}\right)$$

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REVIEW Lecture 18, Cont'd:

- Solution of the Navier-Stokes Equations
  - Pressure Correction Methods:
    - i) Solve momentum for a known pressure leading to new velocity, then
      ii) Solve Poisson to obtain a corrected pressure and
      iii) Correct velocity (and possibly pressure), go to i) for next time-step.
    - A Forward-Euler Explicit (Poisson for p at t_n, then mom. for velocity at t_{n+1})
    - A Backward-Euler Implicit

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(-\frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right) \qquad \frac{\delta}{\delta x_i}\left(\frac{\delta p}{\delta x_i}^{n+1} \right) = \frac{\delta}{\delta x_i}\left(-\frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} \right)$$

  – Nonlinear solvers, Linearized solvers and ADI solvers

- Steady state solvers, implicit pressure correction schemes: iterate using
  – Outer iterations:

$$\mathbf{A}^{\mathbf{u}_i^{m^*}} \mathbf{u}_i^{m^*} = \mathbf{b}^{m-1}_{\mathbf{u}_i^{m^*}} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m-1} \text{ but require } \mathbf{A}^{\mathbf{u}_i^{m}} \mathbf{u}_i^{m} = \mathbf{b}^{m}_{\mathbf{u}_i^{m}} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m} \text{ and } \frac{\boldsymbol{\delta}\, \mathbf{u}_i^m}{\boldsymbol{\delta} x_i} = 0 \quad \Rightarrow \quad 0 \approx \frac{\boldsymbol{\delta}\, \tilde{\mathbf{u}}_i^{m^*}}{\boldsymbol{\delta} x_i} - \frac{\boldsymbol{\delta}}{\boldsymbol{\delta} x_i}\left(\left(\mathbf{A}^{\mathbf{u}_i^{m^*}}\right)^{-1} \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m} \right)$$

  – Inner iterations:

$$\mathbf{A}^{\mathbf{u}_i^{m^*}} \mathbf{u}_i^{m} = \mathbf{b}^{m}_{\mathbf{u}_i^{m^*}} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m}$$

REVIEW Lecture 18, Cont'd:

- Solution of the Navier-Stokes Equations
  - Projection Correction Methods:
    - Construct predictor velocity field that does not satisfy continuity, then correct it using a pressure gradient
    - Divergence producing part of the predictor velocity is "projected out"

- Non-Incremental:
  - No pressure term used in predictor momentum eq.
- Incremental:
  - Old pressure term used in predictor momentum eq.
- Rotational Incremental:
  - Old pressure term used in predictor momentum eq.
  - Pressure update has a rotational correction: $p^{n+1} = p^n + p' = p^n + \delta p^{n+1} + f(u')$

TODAY (Lecture 19)

- Solution of the Navier-Stokes Equations
  - Pressure Correction Methods
    - Projection Methods
      - Non-Incremental, Incremental and Rotational-incremental Schemes
  - Fractional Step Methods:
    - Example using Crank-Nicholson
  - Streamfunction-Vorticity Methods: scheme and boundary conditions
  - Artificial Compressibility Methods: scheme definitions and example
  - Boundary Conditions: Wall/Symmetry and Open boundary conditions
- Time-Time-Marching Methods and ODEs. – Initial Value Problems
  - Euler's method
  - Taylor Series Methods
    - Error analysis
  - Simple 2nd order methods
    - Heun's Predictor-Corrector and Midpoint Method (belong to Runge-Kutta's methods)

References and Reading Assignments

- Chapter 7 on "Incompressible Navier-Stokes equations" of "J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002"
- Chapter 11 on "Incompressible Navier-Stokes Equations" of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, *Computational Fluid Dynamics for Engineers*. Springer, 2005.
- Chapter 17 on "Incompressible Viscous Flows" of Fletcher, *Computational Techniques for Fluid Dynamics*. Springer, 2003.

Projection Methods: Example Scheme 3
Guermond et al, CM-AME-2006

Rotational Incremental (Timmermans et al, 1996):

- Old pressure term used in predictor momentum equation
- Correct pressure based on continuity: $p^{n+1} = p^n + p' = p^n + \delta p^{n+1} + f(u')$
- Update velocity using pressure increment in momentum equation

$$\left(\rho u_i^*\right)^{n+1} = \left(\rho u_i\right)^n + \Delta t\left(-\frac{\delta\left(\rho u_i u_j\right)^{n+1}}{\delta x_j} + \frac{\delta \tau_{ij}^{\;n+1}}{\delta x_j} - \frac{\delta p^n}{\delta x_i}\right) \quad ; \quad \left(\rho u_i^*\right)^{n+1}\Big|_{\partial D} = (\text{bc})$$

$$\tau_{ij}^{\;n+1} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\left.\begin{aligned}\left(\rho u_i\right)^{n+1} &= \left(\rho u_i^*\right)^{n+1} - \Delta t\,\frac{\delta\left(\delta p^{n+1}\right)}{\delta x_i}\\ \frac{\delta\left(\rho u_i\right)^{n+1}}{\delta x_i} &= 0\end{aligned}\right\} \Rightarrow \frac{\delta}{\delta x_i}\left(\frac{\delta\left(\delta p^{n+1}\right)}{\delta x_i}\right) = \frac{1}{\Delta t}\frac{\delta}{\delta x_i}\left(\left(\rho u_i^*\right)^{n+1}\right); \quad \left.\frac{\delta\left(\delta p^{n+1}\right)}{\delta n}\right|_{\partial D} = 0$$

$$\left(\rho u_i\right)^{n+1} = \left(\rho u_i^*\right)^{n+1} - \Delta t\,\frac{\delta\left(\delta p^{n+1}\right)}{\delta x_i}$$

$$p^{n+1} = p^n + \delta p^{n+1} - \mu\,\frac{\delta}{\delta x_i}\left(\left(u_i^*\right)^{n+1}\right)$$

Notes:
- this scheme accounts for ***u***' in the pressure eqn.
- It can be made into a SIMPLE-like method, if iterations are added
- Again, the advection term can be explicit or implicit. The rotational correction to the left assumes explicit advection

Other Methods: Fractional Step Methods

- In the previous methods, pressure is used to:
  - Enforce continuity: it is more a mathematical variable than a physical one
  - Fill the RHS of the momentum eqns. explicitly (predictor step for velocity)
- The fractional step methods (Kim and Moin, 1985) generalize ADI
  - But works on term-by-term (instead of dimension-by-dimension). Hence, does not necessarily use pressure in the predictor step
  - Let's write the NS equations a in symbolic form:

$$u_i^{n+1} = u_i^n + (C_i + D_i + P_i)\,\Delta t$$

where C_i, D_i and P_i represent the convective, diffusive and pressure terms

  - The equation is readily split into a three-steps method:

$$u_i^* = u_i^n + C_i\,\Delta t$$
$$u_i^{**} = u_i^* + D_i\,\Delta t$$
$$u_i^{n+1} = u_i^{**} + P_i\,\Delta t$$

  - In the 3rd step, the pressure gradient ensures u_i^{n+1} satisfy the continuity eq.

Fractional Step Methods, Cont'd

- Many variations of Fractional step methods exists
  - Pressure can be a pseudo-pressure (depends on the specific steps, i.e. what is in u_i^{**}, P_i)
  - Terms can be split further (one coordinate at a time, etc.)
  - For the time-marching, Runge-Kutta explicit, direct 2nd order implicit or Crank-Nicholson scheme are often used
  - Linearization and ADI are also used
  - Used by Choi and Moin (1994) with central difference in space for direct simulations of turbulence (Direct Navier Stokes, DNS)
- Next, we describe a scheme similar to that of Choi and Moin, but using Crank-Nicholson

Fractional Step Methods: Example based on Crank-Nicholson

- In the first step, velocity is advanced using:

$$(\rho u_i)^* - (\rho u_i)^n = \Delta t \left(\frac{H(u_i^n) + H(u_i^*)}{2} - \frac{\delta p^n}{\delta x_i} \right)$$

  - Pressure from the previous time-step
  - Convective, viscous and body forces are represented as an average of old and new values (Crank-Nicolson)
  - Nonlinear equations ⇒ iterate, e.g. Newton's scheme used by Choi et al (1994)
- Second-step: Half the pressure gradient term is removed from u_i^*, to lead u_i^{**}

$$(\rho u_i)^{**} - (\rho u_i)^* = -\Delta t \left(-\frac{1}{2} \frac{\delta p^n}{\delta x_i} \right)$$

- Final step: use half of the gradient of the still unknown new pressure

$$(\rho u_i)^{n+1} - (\rho u_i)^{**} = -\Delta t \left(\frac{1}{2} \frac{\delta p^{n+1}}{\delta x_i} \right)$$

- New velocity must satisfy the continuity equation (is divergence free):
  - Taking the divergence of final step:

$$\frac{\delta}{\delta x_i} \left(\frac{\delta p^{n+1}}{\delta x_i} \right) = \frac{2}{\Delta t} \frac{\delta (\rho u_i)^{**}}{\delta x_i}$$

  - Once p^{n+1} is solved for, the final step above gives the new velocities

Fractional Step Methods: Example based on Crank-Nicholson

- Putting all steps together:

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left[\frac{H(u_i^n) + H(u_i^*)}{2} - \frac{1}{2}\left(\frac{\delta p}{\delta x_i}^n + \frac{\delta p}{\delta x_i}^{n+1} \right) \right]$$

  - To represent Crank-Nicolson correctly, $H(u_i^*)$ should be $H(u_i^{n+1})$
  - However, we can show that the splitting error, $u_i^{n+1} - u_i^*$, is 2nd order in time and thus consistent with C-N's truncation error: indeed, subtract the first step from the complete scheme, to obtain,

$$(\rho u_i)^{n+1} - (\rho u_i)^* = -\frac{\Delta t}{2}\left(\frac{\delta p}{\delta x_i}^{n+1} - \frac{\delta p}{\delta x_i}^n \right) \approx -\frac{\Delta t^2}{2} \frac{\delta}{\delta x_i}\left(\frac{\delta p}{\delta t} \right)$$

  - With this, one also obtains:

$$(\rho u_i)^{n+1} - (\rho u_i)^* = -\frac{\Delta t}{2} \frac{\delta(p^{n+1} - p^n)}{\delta x_i} = -\frac{\Delta t}{2} \frac{\delta(p')}{\delta x_i}$$

    which is similar to the final step, but has the form of a pressure-correction on u_i^*. This later eq. can be used to obtain a Poisson eq. for p' and replace that for p^{n+1}

- Fractional steps methods have become rather popular
  - Many variations, but all are based on the same principles (illustrated by C-N here)
  - Main difference with SIMPLE-type time-marching schemes: SIMPLE schemes solve the nonlinear pressure and momentum equations several times per time-step in outer iterations (iterative nonlinear solve)

Incompressible Fluid
Vorticity Equation

Vorticity

$$\tilde{\omega} \equiv \text{curl}\mathbf{V} \equiv \nabla \times \mathbf{V}$$

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

curl of Navier-Stokes Equation

$$\frac{D\tilde{\omega}}{Dt} = \quad (\tilde{\omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \tilde{\omega}$$

Streamfunction-Vorticity Methods

- For incompressible, 2D flows with constant fluid properties, NS can be simplified by introducing the streamfunction ψ and vorticity ω as dependent variables

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad (\boldsymbol{\omega} = \nabla \times \mathbf{v})$$

  - Streamlines (lines tangent to velocity): constant ψ
  - Vorticity vector is orthogonal to plane of the 2D flow
  - 2D continuity is automatically satisfied: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$!

- In 2D, substituting $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ in $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ leads to the kinematic condition:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

- The vertical component of the vorticity dynamical equation leads:

$$\rho \frac{\partial \omega}{\partial t} + \rho u \frac{\partial \omega}{\partial x} + \rho v \frac{\partial \omega}{\partial y} = \mu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right)$$

Streamfunction-Vorticity Methods, Cont'd

- Main advantages:
  - Pressure does not appear in either of these equations!
  - 2D-NS has been replaced by a set of 2 coupled PDEs
    - Instead of 2 velocities and 1 pressure, we have only two dependent variables
- Explicit solution scheme
  - Given initial velocity field, compute vorticity by differentiation
  - Use this vorticity ω^n in the RHS of the dynamical equation for vorticity, to obtain ω^{n+1}
  - With ω^{n+1} the streamfunction ψ^{n+1} can be obtained from the Poisson equation
    - With ψ^{n+1}, we can differentiate to obtain the velocity
  - Continue to time $n+2$, and so on
- One issue with this scheme: boundary conditions

Streamfunction-Vorticity Methods, Cont'd
Boundary conditions

- Boundary conditions for ψ
  - Solid boundaries are streamlines and require: $\psi = constant$
  - However, values of ψ at these boundaries can be computed only if velocity field is known
- Boundary conditions for ω
  - Neither vorticity nor its derivatives at the boundaries are known in advance
  - For example, at the wall: "$\omega_{\text{wall}} = -\tau_{\text{wall}} / \mu$" since $\tau_{\text{wall}} = \mu \left. \frac{\partial u}{\partial y} \right|_{\text{wall}}$
    - Vorticity at the wall is proportional to the shear stress, but the shear stress is often what one is trying to compute
  - Boundary values for ω can be obtained from $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$,
    - i.e. one-sided differences at the wall: $\frac{\partial^2 \psi}{\partial n^2} = -\omega$

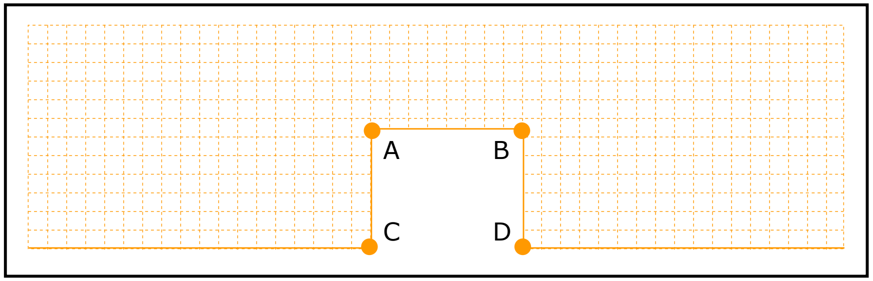
    but this usually converges slowly and can require refinement
  - Discontinuities also occur at corners

Streamfunction-Vorticity Methods, Cont'd

- Discontinuities also occur at corners for vorticity
  - The derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not continuous at A and B
  - This means special treatment for

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

e.g. refine the grid at corners



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- Vorticiy-streamfunction approach useful in 2D, but is now less popular because extension to 3D difficult
  - In 3D, vorticity has 3 components, hence problem becomes as/more expensive as NS
  - Streamfunctiom is still used in quasi-2D problems
    - for example, in the ocean or in the atmosphere, but even there, it has been replaced by level-based models with a free-surface (no steady 2D continuity)

Artificial Compressibility Methods

- Compressible flow is of great importance (e.g. aerodynamics and turbine engine design)
- Many methods have been developed (e.g. MacCormack, Beam-Warming, etc)
- Can they be used for incompressible flows?
- Main difference between incompressible and compressible NS is the mathematical character of the equations
  - Incompressible eqs.: no time derivative in the continuity eqn: $\nabla.\vec{v} = 0$
    - They have a mixed parabolic-elliptic character in time-space
  - Compressible eqs.: there is a time-derivative in the continuity equation:
    - They have a direct hyperbolic character: $$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0$$
    - Allow pressure/sound waves
  - How to use methods for compressible flows in incompressible flows?

Artificial Compressibility Methods, Cont'd

- **Most straightforward:** Append a time derivative to the continuity equation
  - Since density is constant, adding a time-rate-of-change for ρ not possible
  - Use pressure instead (linked to ρ via an eqn. of state in the general case):

$$\frac{1}{\beta}\frac{\partial p}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$

    - where β is an artificial compressibility parameter (dimension of velocity^2)
    - Its value is key to the performance of such methods:
      - The larger/smaller β is, the more/less incompressible the scheme is
      - Large β makes the equation stiff (not well conditioned for time-integration)
    - Methods most useful for solving steady flow problem (at convergence: $\frac{\partial p}{\partial t} = 0$) or inner-iterations in dual-time schemes.
  - To solve this new problem, many methods can be used, especially
    - Time-marching schemes: what we have seen & will see (R-K, multi-steps, etc)
    - Finite differences or finite volumes in space
    - Alternating direction method is attractive: one spatial direction at a time

Artificial Compressibility Methods, Cont'd

- Connecting these methods with the previous ones:
  - Consider the intermediate velocity field $(\rho u_i^*)^{n+1}$ obtained from solving momentum with the old pressure
  - It does not satisfy the incompressible continuity equation: $\frac{\delta(\rho u_i^*)^{n+1}}{\delta x_i} \equiv \frac{\partial \rho^*}{\partial t}$
    - There remains an erroneous time rate of change of mass flux

      ⇒ method needs to correct for it

- Example of an artificial compressibility scheme
  - Instead of explicit in time, let's use implicit Euler (larger time steps for stiff term with large β)

$$\frac{p^{n+1} - p^n}{\beta\, \Delta t} + \left[\frac{\delta(\rho u_i)}{\delta x_i} \right]^{n+1} = 0$$

  - Issue: velocity field at $n+1$ not known ⇒ coupled u_i and p system solve
  - To decouple the system, one could linearize about the old (intermediate) state and transform the above equation into a Poisson equation for the pressure or pressure correction!

Artificial Compressibility Methods: Example Scheme, Cont'd

- Idea 1: expand unknown u_i using Taylor series in pressure derivatives

$$(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta p}\right]^{n+1} (p^{n+1} - p^n) \qquad (p^{*n+1} = p^n)$$

  - Inserting $(\rho u_i)^{n+1}$ in the continuity equation leads an equation for p^{n+1}

$$\frac{p^{n+1} - p^n}{\beta\, \Delta t} + \frac{\delta}{\delta x_i}\left[(\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta p}\right]^{n+1} (p^{n+1} - p^n)\right] = 0$$

  - Expressing $\left[\frac{\delta(\rho u_i^*)}{\delta p}\right]^{n+1}$ in terms of $\frac{\delta p}{\delta x_i}^{n+1}$ using N-S, this is a Poisson-like eq. for p^{n+1}- p^n !

- Idea 2: utilize directly

$$(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta\left(\frac{\delta p}{\delta x_i}\right)}\right]^{n+1} \left(\frac{\delta p}{\delta x_i}^{n+1} - \frac{\delta p}{\delta x_i}^{n}\right)$$

  - Then, still take divergence of $(\rho u_i^*)^{n+1}$ and derive Poisson-like equation

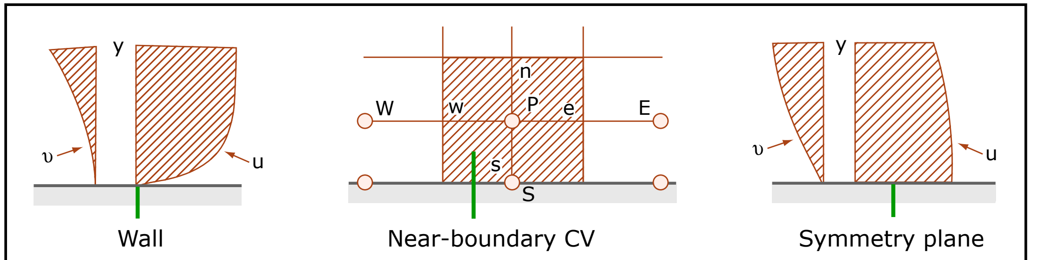
- Ideal value of β is problem dependent
  - The larger the β, the more incompressible. Lowest values of β can be computed by requiring that pressure waves propagate much faster than the flow velocity or vorticity speeds

Numerical Boundary Conditions for N-S eqns.: **Velocity**

- At a wall, the no-slip boundary condition applies:
  - Velocity at the wall is the wall velocity (Dirichlet)
  - In some cases (e.g. fully-developed flow), the tangential velocity is constant along the wall. By continuity, this implies no normal viscous stress:

$$\left.\frac{\partial u}{\partial x}\right|_{\text{wall}} = 0 \;\Rightarrow\; \left.\frac{\partial v}{\partial y}\right|_{\text{wall}} = 0$$

$$\Rightarrow\; \tau_{yy} = 2\mu \left.\frac{\partial v}{\partial y}\right|_{\text{wall}} = 0$$



On the boundary conditions at a wall and a symmetry plane

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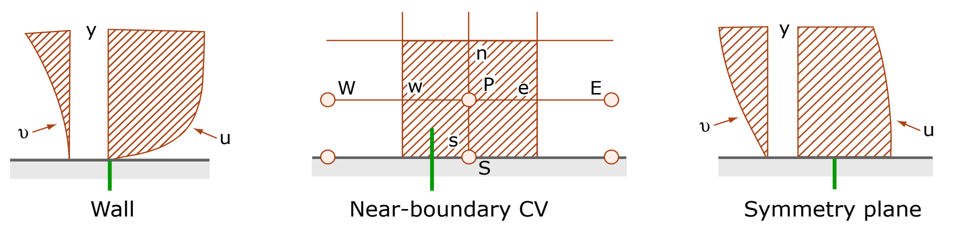
  - For the shear stress: $F_S^{shear} = \int_{S_S} \tau_{xy}\, dS = \int_{S_S} \mu \frac{\partial u}{\partial y}\, dS \approx \mu_S\, S_S \frac{u_P - u_S}{y_P - y_S}$

- At a symmetry plane, it is the opposite:
  - Shear stress is null: $\tau_{xy} = \mu \left.\frac{\partial u}{\partial y}\right|_{\text{sym}} = 0 \;\Rightarrow F_S^{shear} = 0$
  - Normal stress is non-zero:

$$\tau_{yy} = 2\mu \left.\frac{\partial v}{\partial y}\right|_{\text{sym}} \neq 0 \;\Rightarrow F_S^{normal} = \int_{S_S} \tau_{yy}\, dS = \int_{S_S} 2\mu \frac{\partial v}{\partial y}\, dS \approx 2\mu_S\, S_S \frac{v_P - v_S}{y_P - y_S}$$

- Wall/Symmetry Pressure BCs for the Momentum equations
  - For the momentum equations with staggered grids, the pressure is not required at boundaries (pressure is computed in the interior in the middle of the CV or FD cell)
  - With collocated arrangements, values at the boundary for p are needed. They can be extrapolated from the interior (may require grid refinement)
- Wall/Symmetry Pressure BCs for the Poisson equation
  - When the mass flux (velocity) is specified at a boundary, this means that:
    - Correction to the mass flux (velocity) at the boundary is also zero
    - This affects the continuity eq., hence the p eq.: zero normal-velocity-correction $\Rightarrow$ often means gradient of the pressure-correction at the boundary is then also zero

(take the dot product of the velocity correction equation with the normal at the bnd)



On the boundary conditions at a wall and a symmetry plane

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Numerical BCs for N-S eqns: **Outflow/Outlet Conditions**

- Outlet often most problematic since information is advected from the interior to the (open) boundary
- If velocity is extrapolated to the far-away boundary, $\frac{\partial u}{\partial n} = 0 \quad \text{e.g.,} \quad u_E = u_P$,
  - It may need to be corrected so as to ensure that the mass flux is conserved (same as the flux at the inlet)
  - These corrected BC velocities are then kept fixed for the next iteration. This implies no corrections to the mass flux BC, thus a von Neumann condition for the pressure correction (note that p itself is linear along the flow if fully developed).
  - The new interior velocity is then extrapolated to the boundary, etc.
  - To avoid singularities for p (von Neumann at all boundaries for p), one needs to specify p at a one point to be fixed (or impose a fixed mean p)
- If flow is not fully developed: $\frac{\partial u}{\partial n} \neq 0 \quad \Rightarrow \quad \frac{\partial p'}{\partial n} \neq 0 \quad \Rightarrow \quad \text{e.g.} \quad \frac{\partial^2 u}{\partial n^2} = 0 \quad or \quad \frac{\partial^2 p'}{\partial n^2} = 0$
- If the pressure difference between the inlet and outlet is specified, then the velocities at these boundaries can not be specified.
  - They have to be computed so that the pressure loss is the specified value
  - Can be done again by extrapolation of the boundary velocities from the interior: these extrapolated velocities can be corrected to keep a constant mass flux.
- Much research in OBC in ocean modeling

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