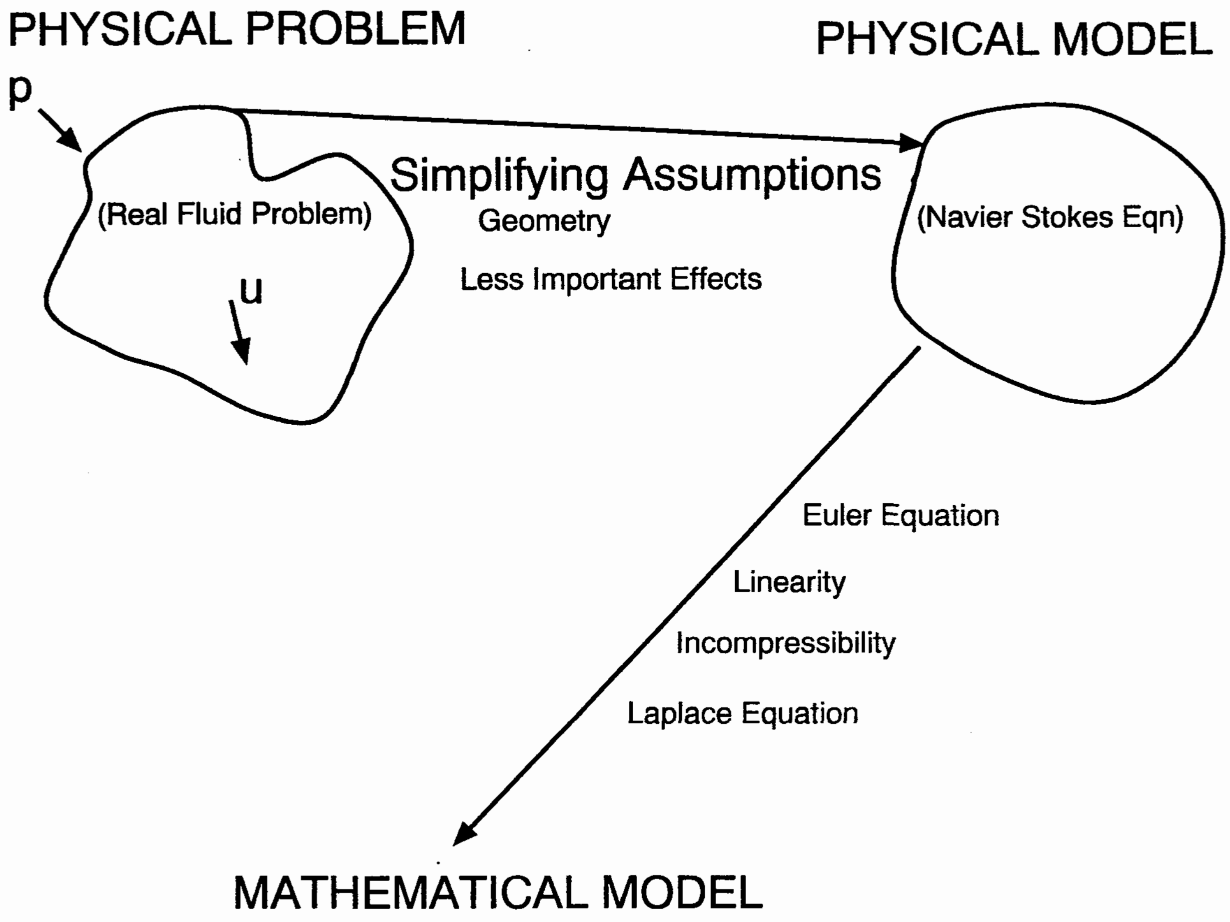
examples_error.pdf

Some Examples and Numerical Errors

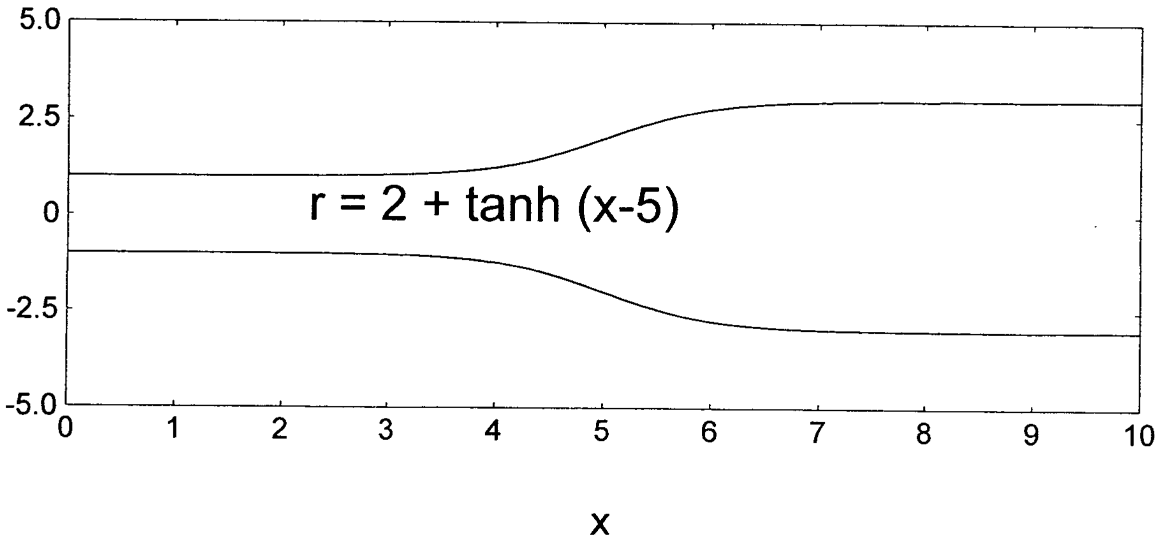
numet1


PHYSICAL PROBLEM
p
(Real Fluid Problem)
u
Simplifying Assumptions
Geometry
Less Important Effects
PHYSICAL MODEL
(Navier Stokes Eqn)
Euler Equation
Linearity
Incompressibility
Laplace Equation
MATHEMATICAL MODEL

Types of Numerical Hydrodynamics Problems

1. Evaluation of Mathematical Functions
2. Simulation
3. Direct Solution of Differential or Integral Equations

Example of Function Evaluation



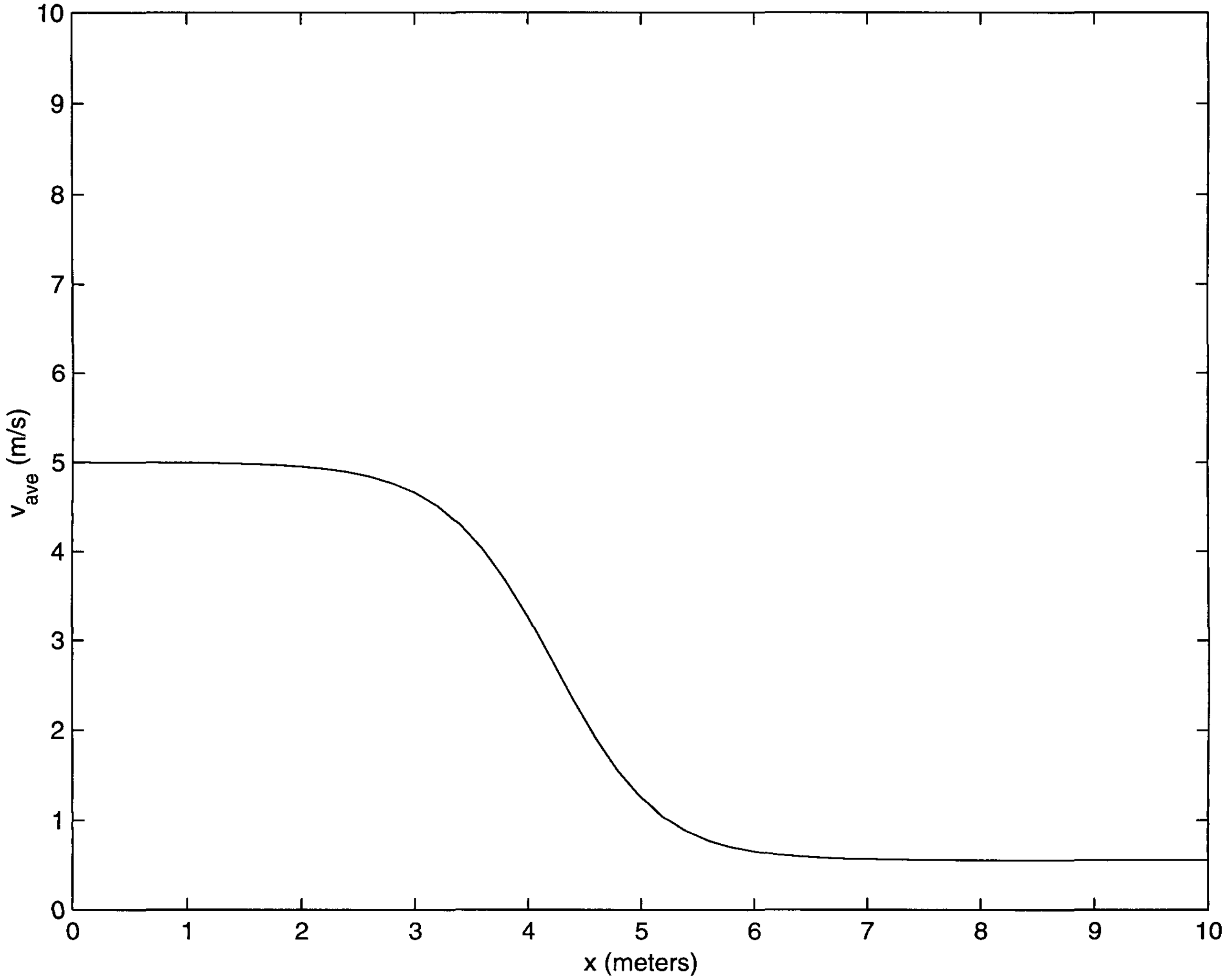
Consider a diffuser with circular crossections and radius vs length as shown. Units are meters. The average velocity of an incompressible fluid across the inlet at $x = 0$ is 5 m/s. Determine the average velocity across all cross sections.

$$Q = 5\pi[2 + \tanh(-5)]^2$$

$$V_{ave}(x) = \frac{Q}{\pi r^2} = \frac{Q}{\pi[2 + \tanh(x - 5)]^2}$$

numet 3

```
% MATLAB Program diffu
fname = input(' Type name for output file: ','s');
fid = fopen(fname,'w');
q = 5.0 * pi * (2.0 + tanh(-5.0))^2;
x = 0 : 0.2 : 10.0 ;
v = q ./ (pi * (2.0 + tanh(x-5.0)) .^2);
for j = 1:51;
    fprintf(fid,'        %10.4f  %10.4f  \n', x(j), v(j) )
end;
fclose(fid);
plot (x,v);
xlabel('x (meters)');
ylabel( 'v_{ave} (m/s)');
axis([0 10 0 10]);
```

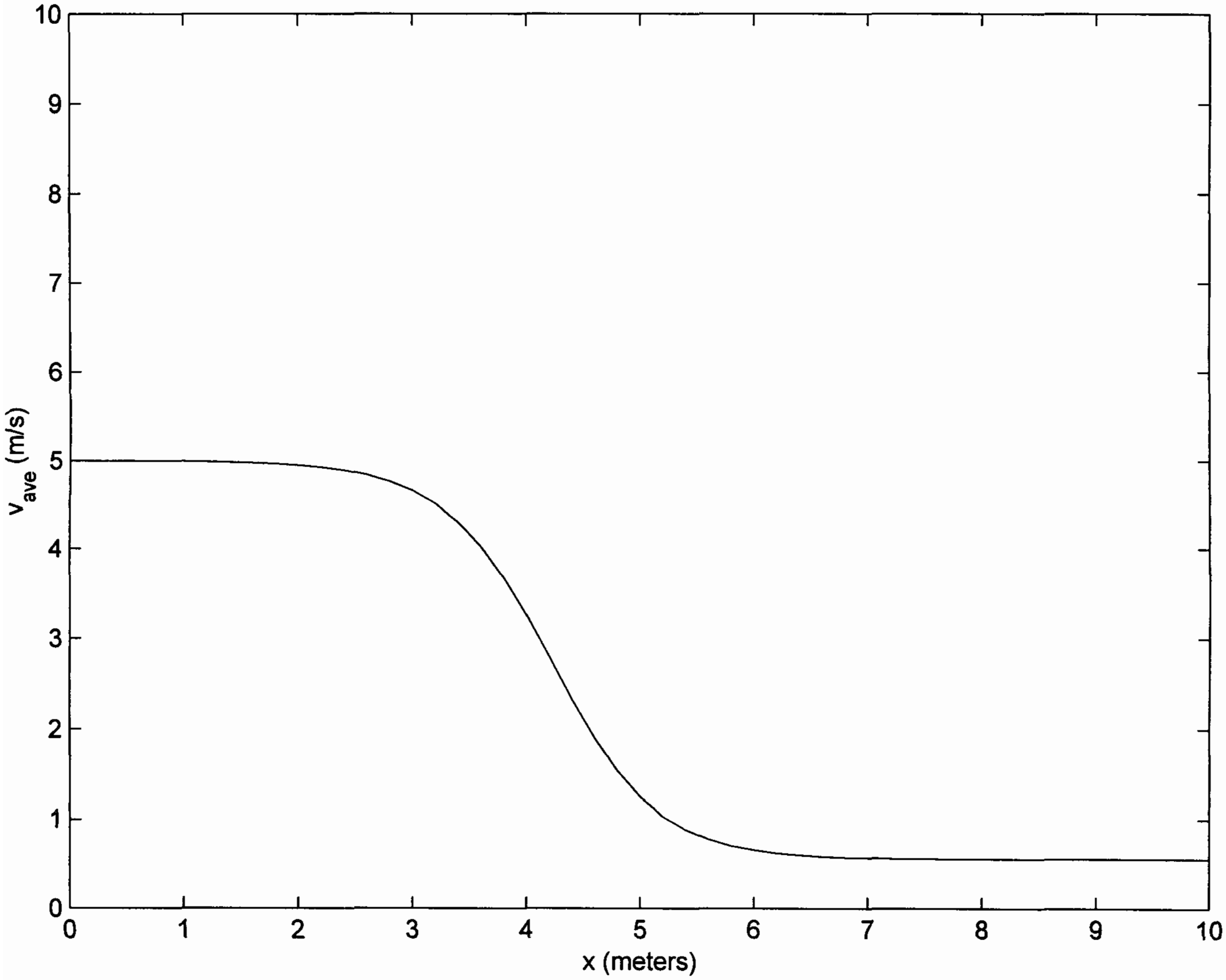


10
9
8
7
6
5
4
3
2
1
0
v_{ave} (m/s)
0
1
2
3
4
5
6
7
8
9
10
x (meters)

numet4

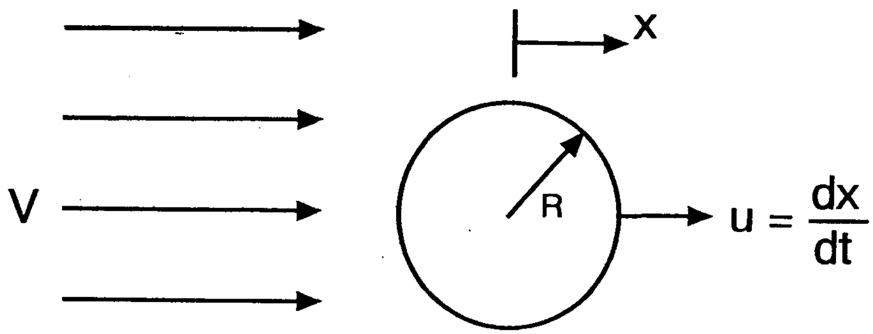
x	v
0.0000	5.0000
0.2000	4.9996
0.4000	4.9989
0.6000	4.9979
0.8000	4.9964
1.0000	4.9942
1.2000	4.9909
1.4000	4.9860
1.6000	4.9787
1.8000	4.9679
2.0000	4.9518
2.2000	4.9280
2.4000	4.8929
2.6000	4.8415
2.8000	4.7668
3.0000	4.6596
3.2000	4.5085
3.4000	4.3008
3.6000	4.0251
3.8000	3.6762
4.0000	3.2608
4.2000	2.8019
4.4000	2.3366
4.6000	1.9054
4.8000	1.5390
5.0000	1.2502
5.2000	1.0357
5.4000	0.8829
5.6000	0.7769
5.8000	0.7046
6.0000	0.6557
6.2000	0.6228
6.4000	0.6007
6.6000	0.5859
6.8000	0.5759
7.0000	0.5692
7.2000	0.5648
7.4000	0.5618
7.6000	0.5597
7.8000	0.5584
8.0000	0.5575
8.2000	0.5569
8.4000	0.5565
8.6000	0.5562
8.8000	0.5560
9.0000	0.5559
9.2000	0.5558
9.4000	0.5558
9.6000	0.5557
9.8000	0.5557
10.0000	0.5557

numet 4a



Example of Solution of Ordinary Differential Equation

Motion of a Sphere Due to Drag



$$M\frac{d^2x}{dt^2} = D \qquad D = \frac{1}{2}\rho C_d \pi R^2 \left(V - \frac{dx}{dt}\right)^2$$

$$\frac{dx}{dt} = u \qquad \frac{du}{dt} = \frac{\rho C_d \pi R^2}{2M}\left(V^2 - 2uV + u^2\right)$$

Discretize and use forward Euler Integration:

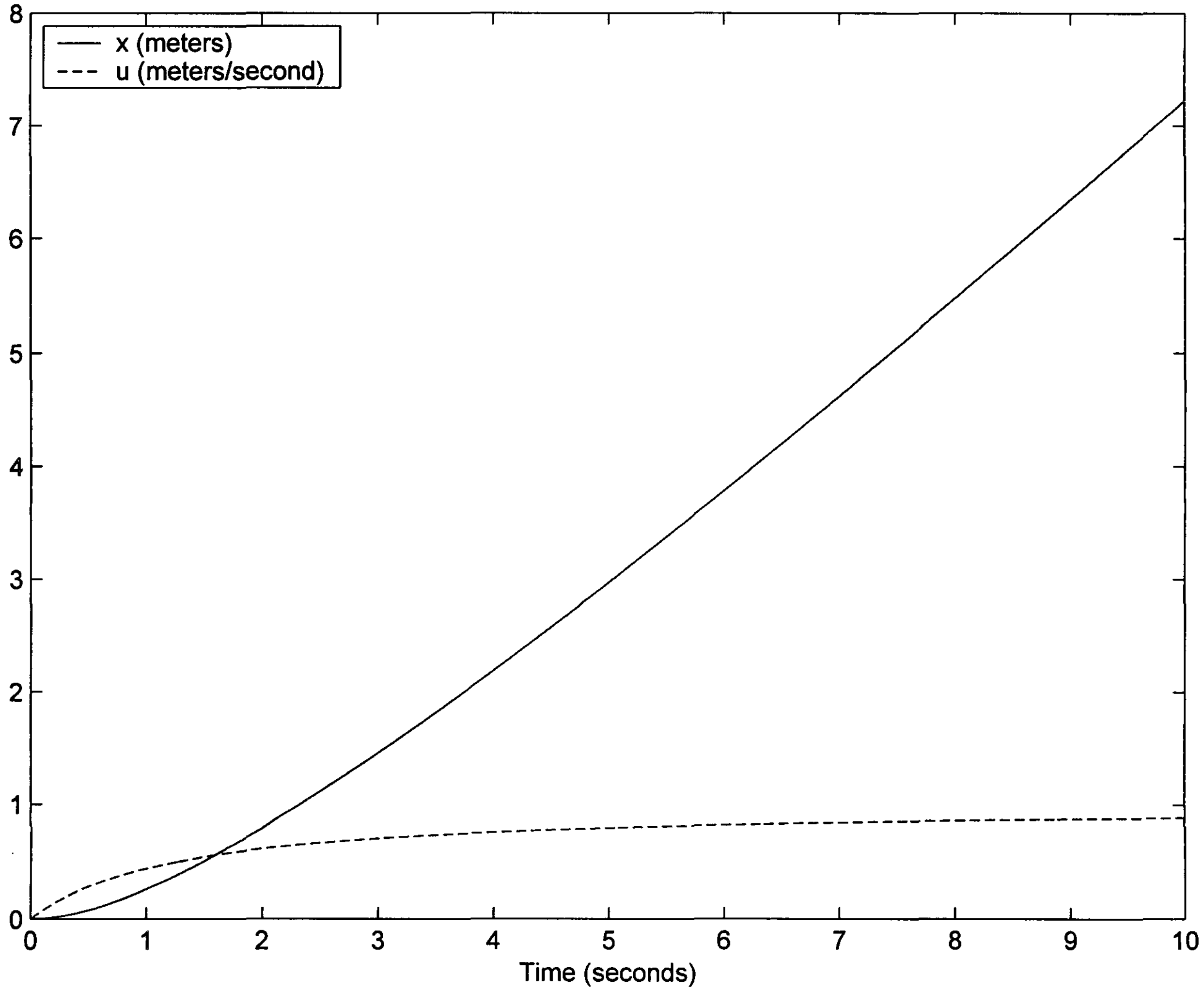
$$u_{i+1} = u_i + \left(\frac{du}{dt}\right)_i \Delta t \qquad x_{i+1} = x_i + u_i \Delta t$$

Initial Conditions: $x_0 = 0$ $u_0 = 0$

```
% MATLAB program spheredg
fid = fopen('sphere.out','w');
rho = 1000.0;
cd = 1.0;
v =1.0;
r = 0.05;
m = 5.0;
fac = rho  * cd *pi *r *r /(2.0 * m);
tt = 10.0;
dt = 0.01;
n = tt/dt + 1;

%initialize

t(1) = 0.0;
x(1) = 0.0;
u(1) = 0.0;
fprintf(fid,'%15.7f   %15.7f   %15.7f  \n', t(1), x(1),
u(1));
for i=2:n;
  j = i-1;
  t(i) = t(j) + dt;
  x(i)= x(j) + u(j)*dt;
  u(i) = u(j) + fac*(v*v- 2.0* u(j)*v + u(j)*u(j))*dt;
  fprintf(fid,'%15.7f   %15.7f   %15.7f \n',t(i),x(i),u(i));
end;
plot(t,x,'-',t,u,'--')
xlabel('Time (seconds)');
h=legend( 'x (meters)' , 'u (meters/second)' , 2);
```


x (meters)
u (meters/second)
0
1
2
3
4
5
6
7
8
0
1
2
3
4
5
6
7
8
9
10
Time (seconds)

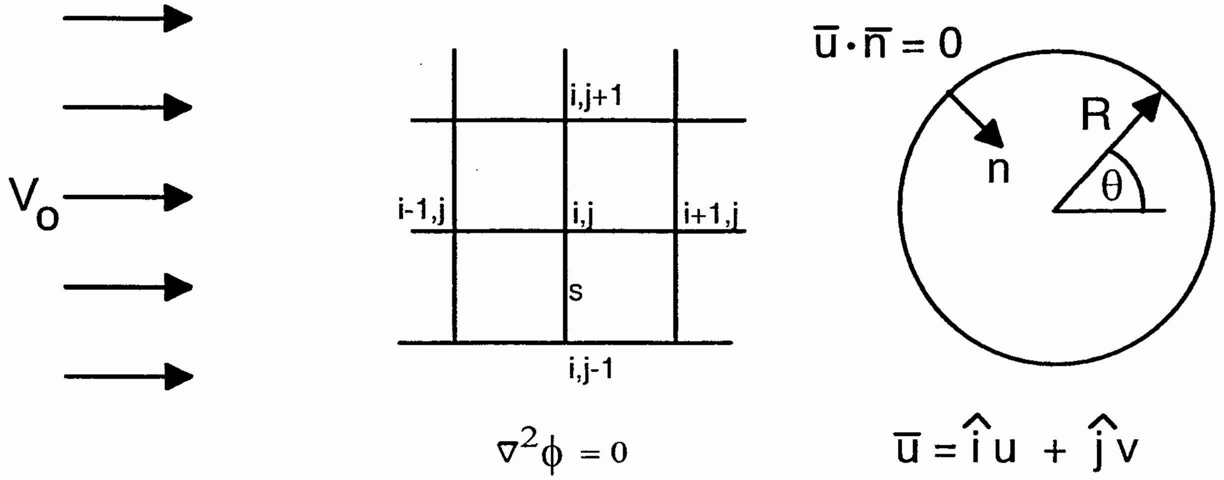
NUMET8B

t	x	u
0.0000000	0.0000000	0.0000000
0.0100000	0.0000000	0.0078540
0.0200000	0.0000785	0.0155851
0.0300000	0.0002344	0.0231962
0.0400000	0.0004664	0.0306900
0.0500000	0.0007733	0.0380693
0.0600000	0.0011539	0.0453367
0.0700000	0.0016073	0.0524947
0.0800000	0.0021323	0.0595457
0.0900000	0.0027277	0.0664922
0.1000000	0.0033926	0.0733364
0.1100000	0.0041260	0.0800807
0.1200000	0.0049268	0.0867271
0.1300000	0.0057941	0.0932779
0.1400000	0.0067269	0.0997350
0.1500000	0.0077242	0.1061005
0.1600000	0.0087852	0.1123762
0.1700000	0.0099090	0.1185642
0.1800000	0.0110946	0.1246662
0.1900000	0.0123413	0.1306840
0.2000000	0.0136481	0.1366193
0.2100000	0.0150143	0.1424739
0.2200000	0.0164391	0.1482493
0.2300000	0.0179215	0.1539472
0.2400000	0.0194610	0.1595691
0.2500000	0.0210567	0.1651166
0.2600000	0.0227079	0.1705911
0.2700000	0.0244138	0.1759940
0.2800000	0.0261737	0.1813267
0.2900000	0.0279870	0.1865906
0.3000000	0.0298529	0.1917871
0.3100000	0.0317708	0.1969174
0.3200000	0.0337399	0.2019828
0.3300000	0.0357598	0.2069844
0.3400000	0.0378296	0.2119236
0.3500000	0.0399489	0.2168014
0.3600000	0.0421169	0.2216190
0.3700000	0.0443331	0.2263776
0.3800000	0.0465968	0.2310781
0.3900000	0.0489076	0.2357217
0.4000000	0.0512648	0.2403094
0.4100000	0.0536679	0.2448422
0.4200000	0.0561163	0.2493210
0.4300000	0.0586096	0.2537469
0.4400000	0.0611470	0.2581207
0.4500000	0.0637282	0.2624434
0.4600000	0.0663527	0.2667159
0.4700000	0.0690198	0.2709390
0.4800000	0.0717292	0.2751137
0.4900000	0.0744804	0.2792406
0.5000000	0.0772728	0.2833207
0.5100000	0.0801060	0.2873548
0.5200000	0.0829795	0.2913435
0.5300000	0.0858929	0.2952877
0.5400000	0.0888458	0.2991882
0.5500000	0.0918377	0.3030455
0.5600000	0.0948682	0.3068606
---	---	---

t	x	u
9.6900000	6.9506656	0.8840910
9.7000000	6.9595065	0.8841966
9.7100000	6.9683485	0.8843019
9.7200000	6.9771915	0.8844070
9.7300000	6.9860356	0.8845120
9.7400000	6.9948807	0.8846167
9.7500000	7.0037269	0.8847213
9.7600000	7.0125741	0.8848256
9.7700000	7.0214224	0.8849298
9.7800000	7.0302717	0.8850338
9.7900000	7.0391220	0.8851376
9.8000000	7.0479734	0.8852413
9.8100000	7.0568258	0.8853447
9.8200000	7.0656792	0.8854479
9.8300000	7.0745337	0.8855510
9.8400000	7.0833892	0.8856539
9.8500000	7.0922458	0.8857566
9.8600000	7.1011033	0.8858591
9.8700000	7.1099619	0.8859614
9.8800000	7.1188215	0.8860635
9.8900000	7.1276822	0.8861655
9.9000000	7.1365438	0.8862673
9.9100000	7.1454065	0.8863689
9.9200000	7.1542702	0.8864703
9.9300000	7.1631349	0.8865715
9.9400000	7.1720006	0.8866725
9.9500000	7.1808673	0.8867734
9.9600000	7.1897351	0.8868741
9.9700000	7.1986038	0.8869746
9.9800000	7.2074735	0.8870749
9.9900000	7.2163443	0.8871751
10.0000000	7.2252160	0.8872751

Example of Solution of Partial Differential Equation

Potential Streaming Flow about a Circular Cylinder



There is an equation at each grid point:

$$\frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}}{s^2} + \frac{\phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{s^2} = 0$$

Boundary Condition far (many diameters) from the cylinder:

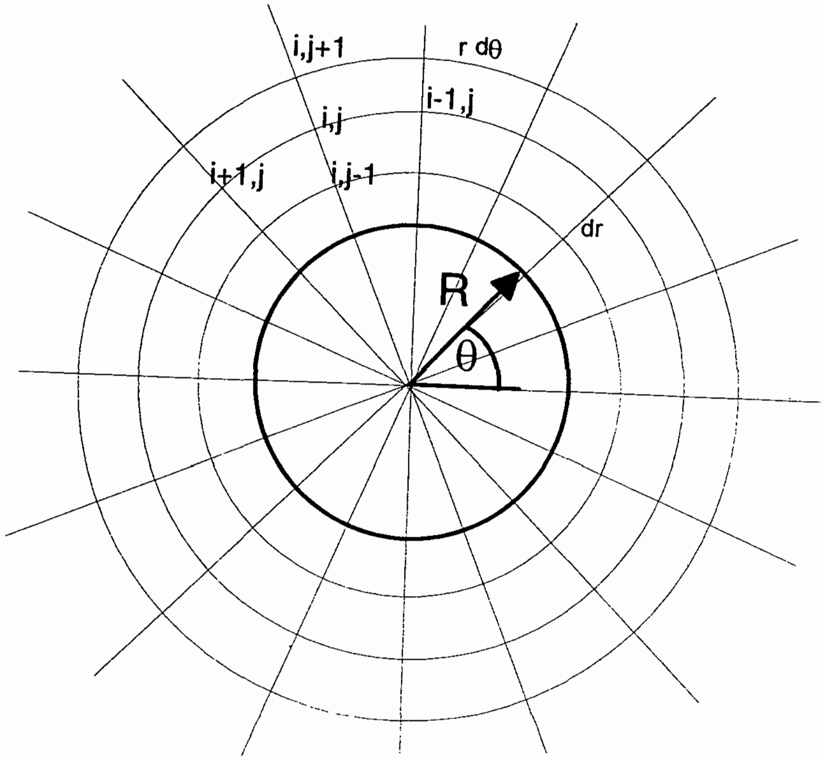
$$\frac{\phi_{i+1,j} - \phi_{i,j}}{s} = V_o \qquad \frac{\phi_{i,j+1} - \phi_{i,j}}{s} = 0$$

Boundary Condition on the cylinder:

$$\frac{\phi_{i+1,j} - \phi_{i,j}}{s}\cos\theta + \frac{\phi_{i,j+1} - \phi_{i,j}}{s}\sin\theta = 0$$

NUmet 10

cylindrical coordinates



Equation at each grid point:

$$\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2rdr} + \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(dr)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{r^2 d\theta^2} = 0$$

Boundary Condition far (many diameters) from the cylinder:

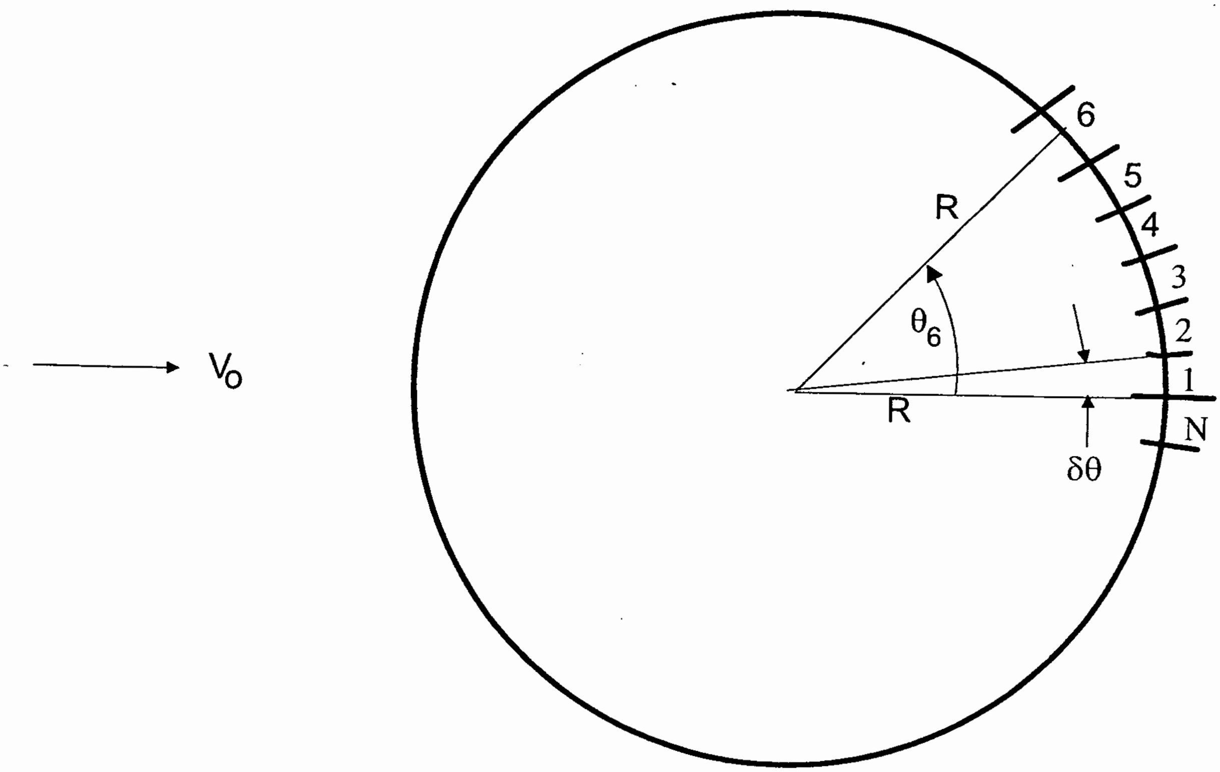
$$\frac{\phi_{i,j+1} - \phi_{i,j}}{dr} = V_o \cos\theta \qquad \frac{\phi_{i+1,j} - \phi_{i,j}}{rd\theta} = -V_o \sin\theta$$

Boundary Condition on the cylinder:

$$\frac{\phi_{i,j+1} - \phi_{i,j}}{dr} = 0$$

Example of Discretized Integral Equation

Potential Streaming Flow about a Circular Cylinder Here we seek the perturbation velocity that exists in addition to the uniform flow of speed V_o.



Exterior flow is represented as a source distribution of strength $\sigma(\theta)$ on the surface of the cylinder.

$$\vec{U}(r) = \int_0^{2\pi} \frac{\sigma(\theta)[r - \vec{r}'(\theta)]}{2\pi \left| r - \vec{r}'(\theta) \right|^2} R d\theta$$

Just outside the surface of the cylinder, $\vec{U} \cdot \vec{n} = -V_o \cos\theta$

Now we can form the discretized approximate equation:

$$\sum_{i=1, i \neq j}^{N} \frac{\sigma_i (r_j \vec{-} r_i) \cdot \vec{n_j}}{2\pi |r_j - r_i|^2} R\delta\theta + \frac{\sigma_j}{2} = -V_o \cos\theta_j$$

Stability

When applying a numerical procedure to a problem in fluid mechanics, the result can diverge. In other words, the process is unstable. Such instabilities can be fundamentally fluid mechanical or they may come from inaccuracies in the numerical procedure.

For example, suppose a process is governed by the differential equation:

$$\frac{dy}{dt} = 3y \quad \text{with initial condition} \qquad y(0) = 1$$

We know that the solution to this equation is $y = e^{3t}$ which diverges as t increases. This is a fundamental instability in the process being modeled. A proper numerical solution will capture this instability.

Now we explore a numerical instability using numerical values with three decimal places. Consider the set of values, z_n defined for non-negative integers n by:

$$z_n = \int_0^1 \frac{x^n}{x+5} dx$$

A recursion relation for the z's can be made as follows:

$$z_n + 5z_{n-1} = \int_0^1 \frac{x^n + 5x^{n-1}}{x+5} dx = \int_0^1 \frac{x^{n-1}(x+5)}{x+5} dx = \frac{1}{n}$$

$$z_n = \frac{1}{n} - 5\, z_{n-1}$$

$$\text{For n} = 0, \quad z_o = \int_0^1 \frac{1}{x+5} dx = \ln(6) - \ln(5) = 0.182$$

$$\text{For n} > 0, \quad z_n = \frac{1}{n} - 5\, z_{n-1}$$

$$z_1 = 1.000 - 5 \times 0.182 = 0.090$$

$$z_2 = 0.500 - 5 \times 0.090 = 0.050$$

$$z_3 = 0.333 - 5 \times 0.050 = 0.083$$

$$z_4 = 0.250 - 5 \times 0.083 = -0.165$$

The above negative value for z_4 must be wrong since the integrand is positive. It comes from numerical instability associated with roundoff error.

An alternative recursion relation is:

$$z_{n-1} = \frac{1}{5n} - \frac{Z_n}{5} = 0.2\left(\frac{1}{n} - z_n\right)$$

This reduces the effect of the error by a factor of 5. We will start with an approximation to z_{10} and use the recursion relation for successively smaller values of n.

Approximate Equation: $z_{10} = \int_0^1 \frac{x^{10}}{5(1+0.2x)} \approx 0.2 \int_0^1 x^{10}(1-0.2x)dx = 0.015$

$$z_{10} = 0.015$$

$$z_9 = 0.2\left(\frac{1}{10} - 0.015\right) = 0.017$$

$$z_8 = 0.2\left(\frac{1}{9} - 0.017\right) = 0.019$$

$$z_7 = 0.2\left(\frac{1}{8} - 0.019\right) = 0.021$$

$$z_6 = 0.2\left(\frac{1}{7} - 0.021\right) = 0.024$$

$$z_5 = 0.2\left(\frac{1}{6} - 0.024\right) = 0.029$$

$$z_4 = 0.2\left(\frac{1}{5} - 0.029\right) = 0.034$$

$$z_3 = 0.2\left(\frac{1}{4} - 0.034\right) = 0.043$$

$$z_2 = 0.2\left(\frac{1}{3} - 0.043\right) = 0.058$$

$$z_1 = 0.2\left(\frac{1}{2} - 0.058\right) = 0.088$$

$$z_0 = 0.2\left(\frac{1}{1} - 0.088\right) = 0.182$$

Note that the result for z_0 is cortrect even though an approximate value was used for Z_{10}. This iteration scheme is stable.