The Generalized Newtonian Fluid - Isothermal Flows

- Constitutive Equations
- Viscosity Models
- Solution of Flow Problems

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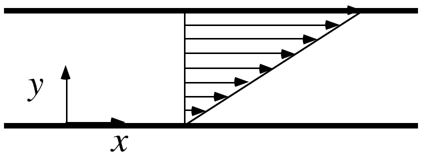
Generalized Newtonian Fluid

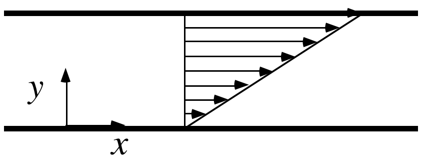
Simple Shear Flow

Newtonian Fluid



$$\tau_{yx} = -\mu \frac{dv_x}{dy}$$

Non-Newtonian Fluid



$$\tau_{yx} = -\eta \frac{dv_x}{dy}$$

where

$$\eta = \eta\left(\left|dv_x / dy\right|\right)$$

Arbitrary Flow

Newtonian Fluid

$$\tau_{ij} = -\mu\left(\frac{dv_j}{dx_i} \quad \frac{dv_i}{dx_j}\right)$$

$$= -\mu\dot{\gamma}_{ij}$$

Non-Newtonian Fluid

$$\tau_{ij} = -\eta\dot{\gamma}_{ij}$$

where

$$\eta = \eta(\dot{\gamma})$$

$$\dot{\gamma} = \sqrt{\tfrac{1}{2}\mathrm{II}_{\dot{\gamma}}}$$

Non-Newtonian Viscosity

- η is a scalar
- It must depend only on scalar invariants of $\dot{\boldsymbol{\gamma}}$
- Three scalar invariants are defined for $\dot{\boldsymbol{\gamma}}$

$$\mathrm{I}_{\dot{\gamma}} = \sum_i \dot{\gamma}_{ii} = 2(\nabla \cdot \mathbf{v}) = 0$$

$$\mathrm{II}_{\dot{\gamma}} = \sum_i \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{ji} = 2\dot{\gamma}$$

$$\mathrm{III}_{\dot{\gamma}} = \sum_i \sum_j \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{jk} \dot{\gamma}_{ki} = 0$$

(last equality is for shear flow)

- Since II is the only non-zero invariant in simple shear flow, define

$$\dot{\gamma} = \sqrt{\tfrac{1}{2}\mathrm{II}_{\dot{\gamma}}}$$

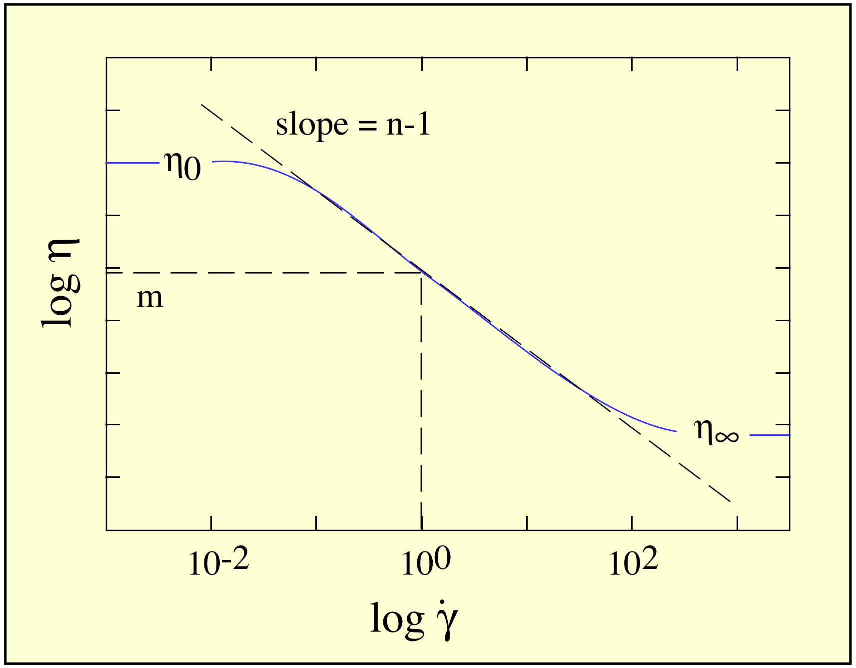
definition of shear rate

Generalized Newtonian Fluid

- Restrictions
    - Does not describe normal stress phenomena
    - Does not describe time-dependent phenomena
    - Strictly speaking, it applies only to **shear stresses** in **steady shear flow**
- Empiricisms for $\eta(\dot{\gamma})$
    - Power-law model
    - Spriggs truncated power-law model
    - Carreau model
    - Bingham model
    - Casson model

Power-Law Model

$$\eta = m\dot{\gamma}^{n-1}$$



- Captures high shear rate behavior normally occuring in processing

$n < 1$	shear thinning $[0.15 < n < 0.6]$
$n = 1$	Newtonian fluid $m = \mu$
$n > 1$	shear thickening

- Defects
  - No η_0
  - No λ

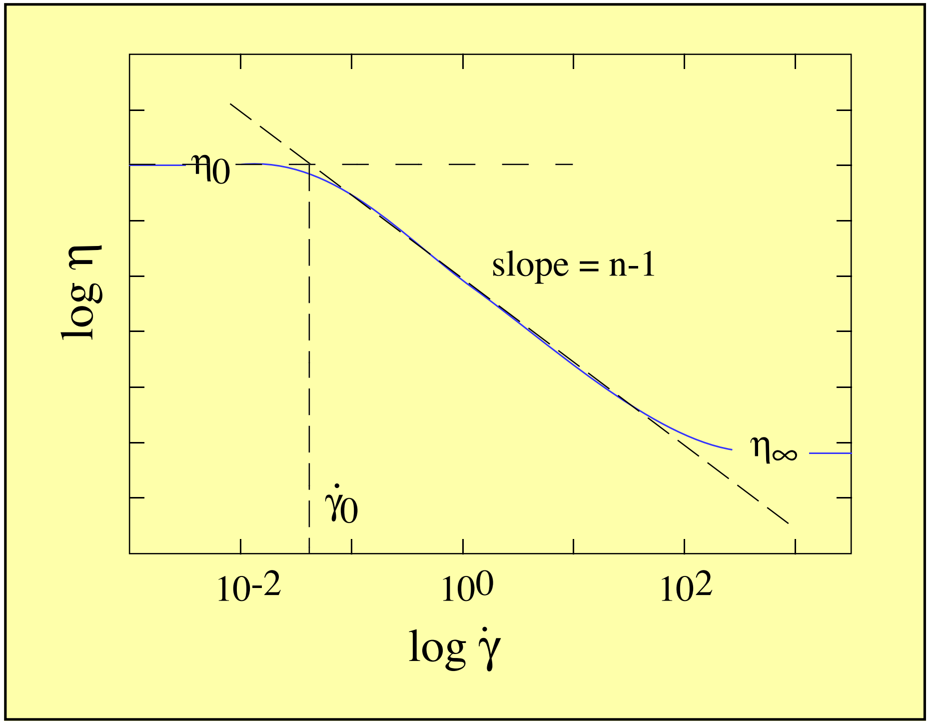
Spriggs "Truncated Power-Law"

$$\eta = \eta_0 \qquad \dot{\gamma} \le \dot{\gamma}_0$$

$$\eta = \eta_0 \left(\frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{n-1} \qquad \dot{\gamma} > \dot{\gamma}_0$$

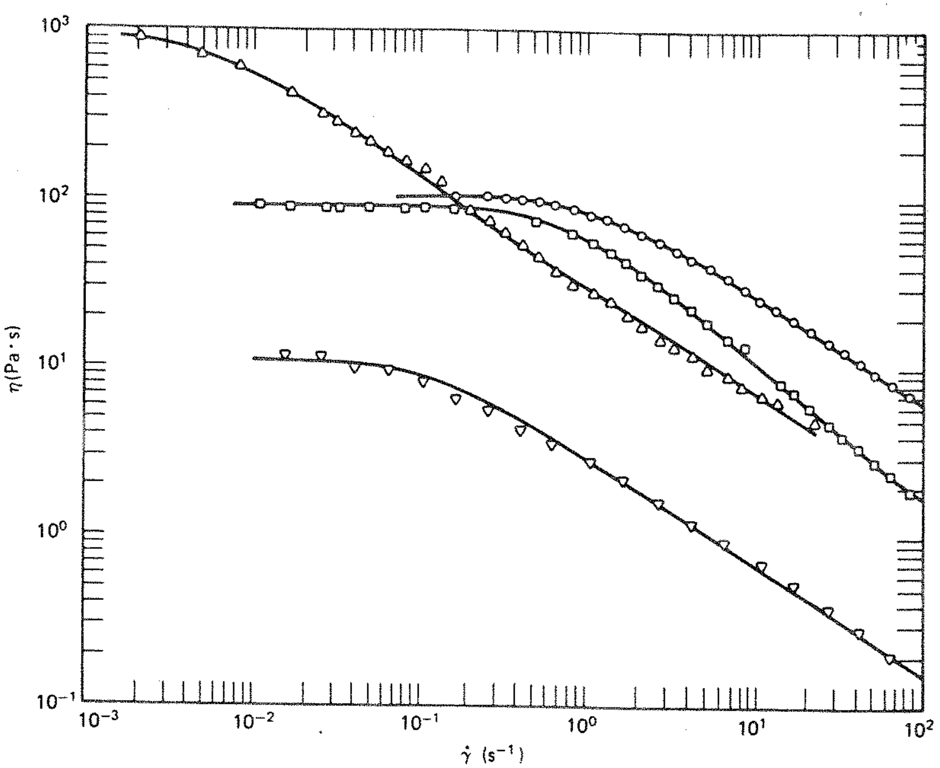
- Contains a characteristic time λ

$$\lambda = \frac{1}{\dot{\gamma}_0}$$



Carreau-Yasuda Model

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = \left[1 + (\lambda\dot{\gamma})^a\right]^{\frac{n-1}{a}}$$



η (Pa·s)

$\dot{\gamma}$ (s^{-1})

Bingham Model

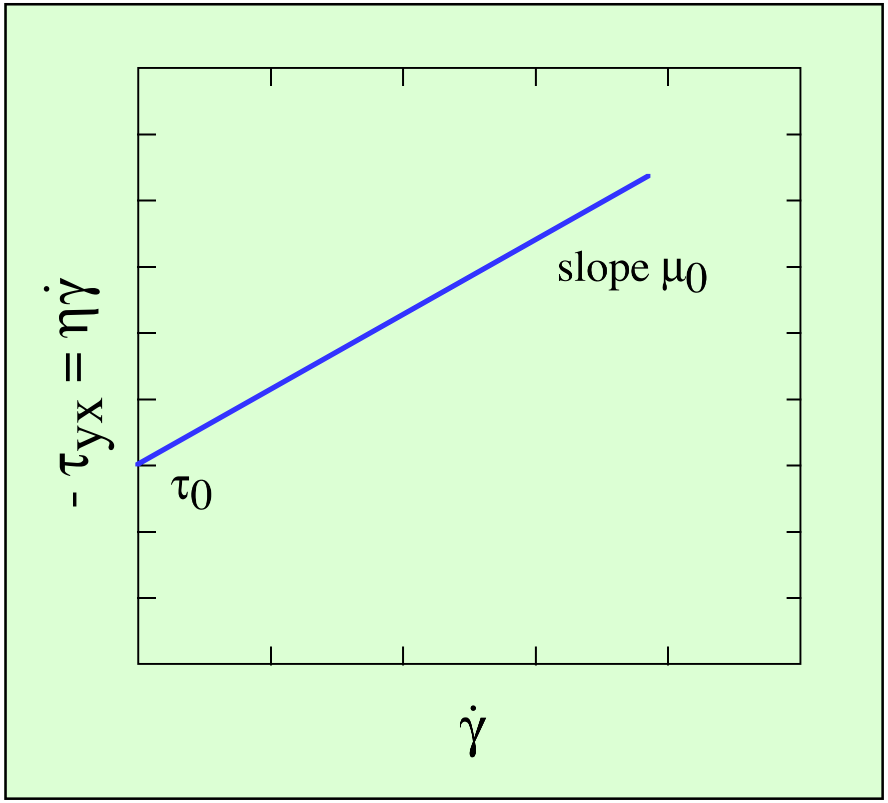
$$\eta = \begin{cases} \infty & (\tau \le \tau_0) \\ \mu_0 + \dfrac{\tau_0}{\dot{\gamma}} & (\tau \ge \tau_0) \end{cases}$$

- Bingham plastics have a yield stress τ_0 and will not flow unless the magnitude of the stress τ exceeds τ_0

$$\tau = \sqrt{\tfrac{1}{2}(\boldsymbol{\tau} : \boldsymbol{\tau})}$$

- Time constant

$$\frac{\mu_0}{\tau_0}$$



Casson Model

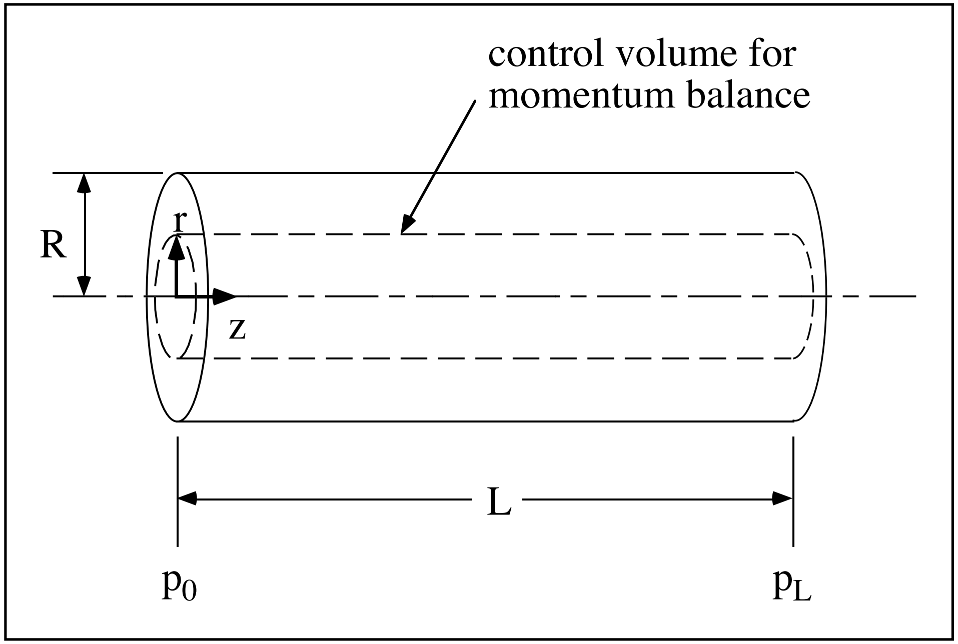
$$\sqrt{\pm\tau_{yx}} = \sqrt{\tau_0} + \sqrt{\mu_0}\sqrt{\mp\frac{dv_x}{dy}} \qquad \text{for } \tau_{yx} > \tau_0$$

$$\dot{\gamma}_{yx} = 0 \qquad \text{for } \tau_{yx} < \tau_0$$

- Useful for chocolate

Tube Flow of a Power-Law Fluid

- Force balance on cylindrical control volume of length L and radius r



$$\left(p_0 - p_L\right)\pi r^2 - \tau_{rz} \cdot 2\pi r L = 0$$

$$\tau_{rz} = \tau_w \frac{r}{R}$$

$$\tau_w = \frac{\Delta p\, R}{2L}$$

- Constitutive equation gives a second expression for τ_{rz}

$$\tau_{rz} = -\eta(\dot{\gamma})\dot{\gamma}_{rz} = -m\dot{\gamma}^{n-1}\frac{dv_z}{dr}$$

$$\tau_{rz} = m\left(-\frac{dv_z}{dr}\right)^n$$

Tube Flow Results

- Velocity profile

$$v_z = \left(\frac{\tau_w}{m}\right)^{\frac{1}{n}} \frac{R}{\frac{1}{n}+1}\left[1-\left(\frac{r}{R}\right)^{\frac{1}{n}+1}\right]$$

- Volume flow rate

$$Q = \frac{\pi R^3}{\frac{1}{n}+3}\left[\frac{\tau_w}{m}\right]^{\frac{1}{n}}$$

- From v_z calculate

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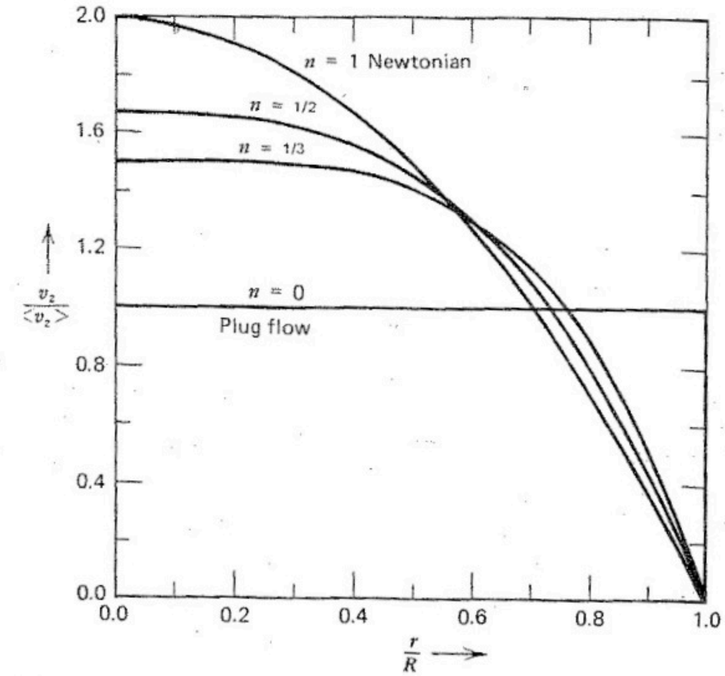
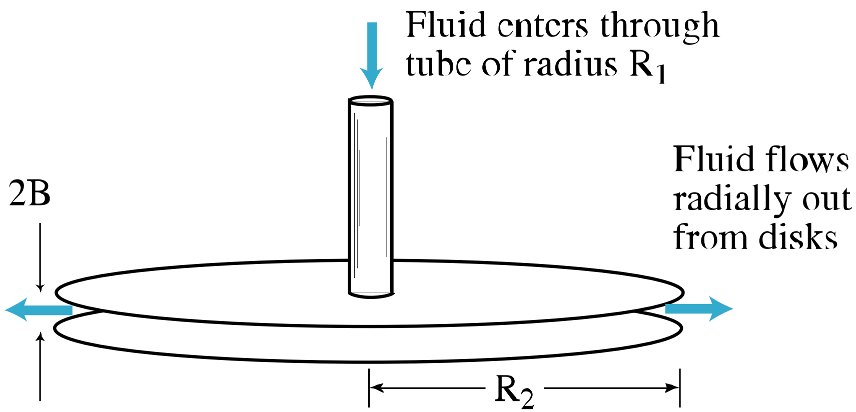


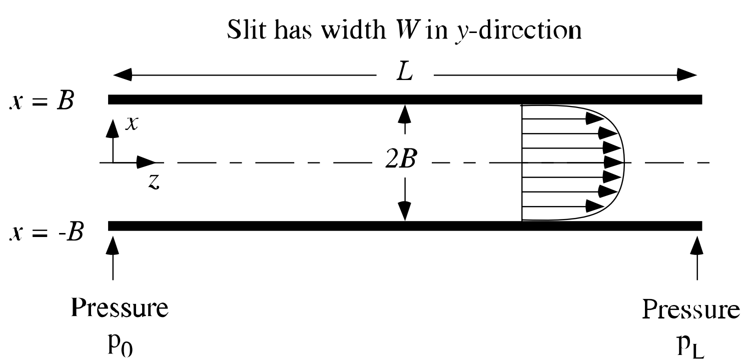
FIGURE 4.2-1. Tube flow velocity profiles for a power-law fluid from Eq. 4.2-8. Note that the profiles become increasingly flatter as n decreases; $n = 0$ corresponds to plug flow. The Newtonian (parabolic) profile is shown as $n = 1$.

$$\dot{\gamma}_R = \left(\frac{3n+1}{4n}\right)\dot{\gamma}_a \qquad \dot{\gamma}_a = \frac{8\langle v_z\rangle}{D}$$

true wall shear rate — apparent (Newtonian) wall shear rate

Radial Flow between Parallel Disks





- Problem:
  - Determine $Q(p_1\text{-}p_2, B, R_1, R_2, m, n)$
  - Assume the fluid viscosity is described by the power-law function
- Solution:
  - Use the lubrication approximation to simplify the problem to pressure driven slit flow
  - First find Q for this simple flow

- Assume

$$v_z = v_z(x), \quad v_x = v_y = 0$$

$$\tau_{ij} = \tau_{ij}(x)$$

- z-component of the equation of motion (Table B.1)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] - \frac{\partial p}{\partial z} + \rho g_z$$

$$\frac{d}{dx}\tau_{xz} = \frac{p_0 - p_L}{L}$$

- Integrate to get the shear stress distribution from conservation of momentum

$$\tau_{xz} = \frac{p_0 - p_L}{L} x + c_1$$

0

- For the generalized Newtonian fluid

$$\tau_{xz} = -\eta \frac{dv_z}{dx}$$

and the power-law model gives

$$\eta = m\dot{\gamma}^{n-1} = m\left|\frac{dv_z}{dx}\right|^{n-1}$$

- In order to avoid problems with the absolute value, consider only the region x > 0, for which

$$\frac{dv_z}{dx} < 0 \quad \Rightarrow \quad \left|\frac{dv_z}{dx}\right| = \left(-\frac{dv_z}{dx}\right)$$

- Then

$$\tau_{xz} = -m\left(-\frac{dv_z}{dx}\right)^{n-1} \frac{dv_z}{dx} = m\left(-\frac{dv_z}{dx}\right)^{n} = (p_0 - p_L)\frac{x}{L}$$

- The differential equation for the velocity is thus

$$-\frac{dv_z}{dx} = \left(\frac{p_0 - p_L}{mL}\right)^{1/n} x^{1/n}$$

which can be integrated to give

$$v_z = \left(\frac{p_0 - p_L}{mL}\right)^{1/n} \frac{1}{\frac{1}{n}+1}\left(B^{\frac{1}{n}+1} - x^{\frac{1}{n}+1}\right)$$

- Finally, the volume flow rate is found as

$$Q = 2W\int_0^B v_z\,dx = \frac{2WB^2}{\frac{1}{n}+2}\left(\frac{\tau_B}{m}\right)^{1/n}$$

where

$$\tau_B = \frac{p_0 - p_L}{L}B = -\frac{dp}{dz}B$$

Apply the slit flow results locally to the disk problem

- The corresponding quantities in the two geometries are
  - pressure gradient

$$\frac{p_0 - p_L}{L} = -\frac{dp}{dz} \quad \rightarrow \quad -\frac{dp}{dr}$$

  - width

$$W = 2\pi r$$

- The volume flow rate expression is adapted as

$$\left[\frac{\left(\frac{1}{n} + 2\right) Q}{2B^2} \frac{1}{2\pi r} \right]^n \frac{m}{B} = -\frac{dp}{dr}$$

which can be integrated from $r = R_1$ to $r = R_2$, by taking advantage of the fact that **for incompressible fluids *Q* is independent of *r***. This gives

$$p_1 - p_2 = \left[\frac{\left(\frac{1}{n} + 2\right) Q}{4\pi B^2} \right]^n \frac{m}{B} \frac{\left(R_2^{1-n} - R_1^{1-n}\right)}{(1-n)}$$

- Finally, the above is inverted to give the volume flow rate in terms of the pressure gradient

$$Q = \frac{4\pi B^2}{\left(\frac{1}{n} + 2\right)} \left[\frac{\left(p_1 - p \quad\right) B(1-n)}{m\left(R_2^{1-n} - R_1^{1-n}\right)} \right]^{1/n}$$

Justification for Applying the Lubrication Approximation

Use an order of magnitude analyis to justify the use of the lubrication approximation in adapting the slit flow results to the radial disk flow problem

- For the radial flow problem, assume that

$$v_r = v_r(r, z), \quad v_\theta = v_z = 0$$

- r-component of the equation of motion

$$\rho v_r \frac{\partial v_r}{\partial r} = -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] - \frac{\partial p}{\partial r}$$

- For the power-law fluid

- For the power-law fluid

$$\tau_{ij} = -m\dot{\gamma}^{n-1}\dot{\gamma}_{ij}; \qquad \dot{\gamma} = \sqrt{\tfrac{1}{2}\left(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}}\right)}$$

- From Table B.3 in DPL, we get the rate-of-strain tensor in cylindrical coordinates

$$\dot{\boldsymbol{\gamma}} = \begin{pmatrix} 2\dfrac{\partial v_r}{\partial r} & 0 & \dfrac{\partial v_r}{\partial z} \\ 0 & 2\dfrac{v_r}{r} & 0 \\ \dfrac{\partial v_r}{\partial z} & 0 & 0 \end{pmatrix}$$

from which the shear rate is found

$$\dot{\gamma} = \sqrt{2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial v_r}{\partial z}\right)^2}$$

- Order of magnitude estimates for contributions to the shear rate

$$v_r \sim V = \frac{Q}{4\pi R_1 B}$$

$$\frac{\partial v_r}{\partial r} \sim \frac{V}{R_2}; \qquad \frac{v_r}{r} \sim \frac{V}{R_2}; \qquad \frac{\partial v_r}{\partial z} \sim \frac{V}{B}$$

for $R_1 << R_2$

- For small gap, $B << R_2$, the shear rate is well approximated by

$$\dot{\gamma} \sim \left| \frac{\partial v_r}{\partial z} \right| \sim \frac{V}{B}$$

- Next evaluate the order of magnitude of the terms that appear in the equation of motion

(Inertial term) $$\rho v_r \frac{\partial v_r}{\partial r} \sim \rho \frac{V^2}{R_2}$$

(Stress terms)

$$\frac{1}{r}\frac{\partial}{\partial r} r\tau_{rr} \sim \frac{mV^n}{R_2^2 B^{n-1}}; \qquad \frac{\partial}{\partial z}\tau_{zr} \sim \frac{m}{B}\left(\frac{V}{B}\right)^n; \qquad \frac{\tau_{\theta\theta}}{r} \sim \frac{m}{R_2}\left(\frac{V}{B}\right)^{n-1}\frac{V}{R_2}$$

- Comparison of different terms shows that $\partial\tau_{zr}/\partial z$ is the largest term on the right side by a factor of

$$\left(B/R_2\right)^2 \ll 1$$

- The ratio of inertial to viscous forces is

$$\frac{\rho\dfrac{V^2}{R_2}}{\dfrac{m}{B}\left(\dfrac{V}{B}\right)^n} = \frac{\rho V R_2}{m\left(V/B\right)^{n-1}}\cdot\left(\frac{B}{R_2}\right)^2 = \mathrm{Re}\cdot\left(\frac{B}{R_2}\right)^2$$

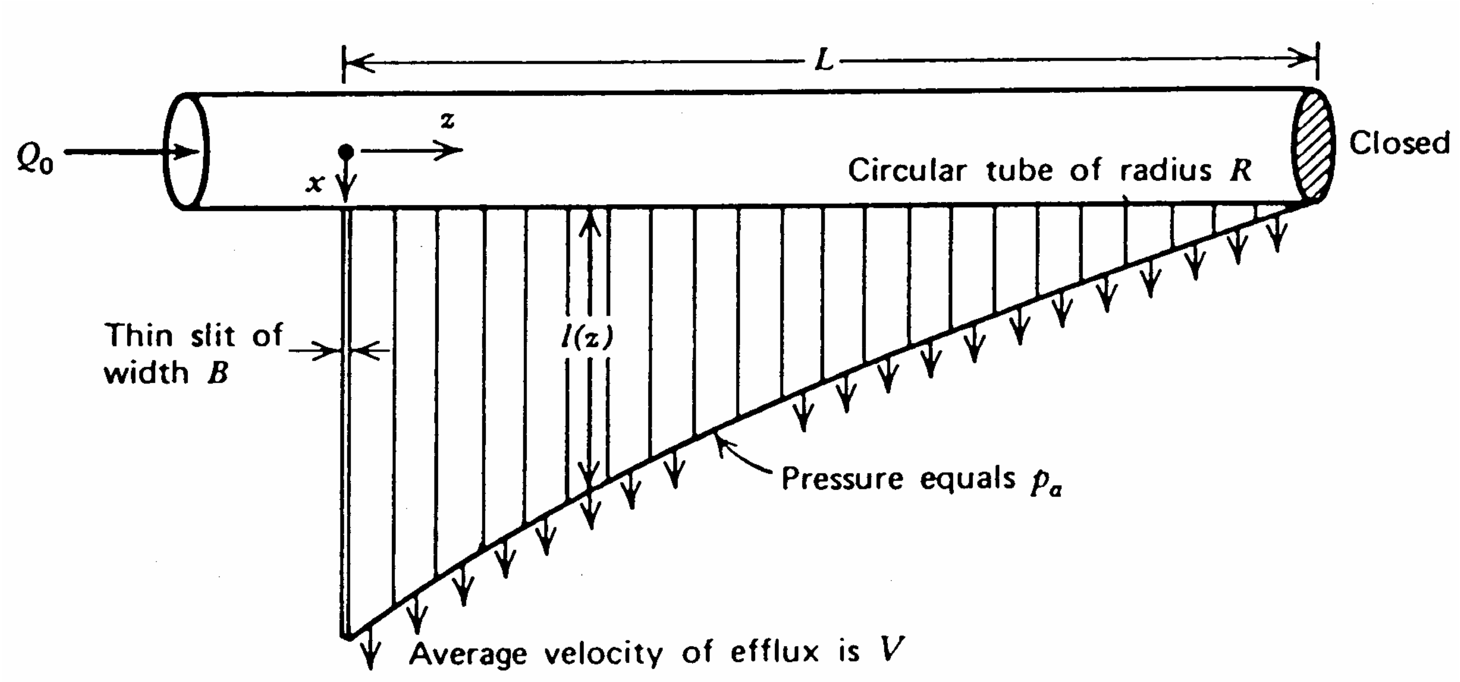
- If the Reynolds number is at most (R_2/B), then inertial forces can still be neglected

- Neglecting terms of order (B/R_2) and smaller gives the equation of motion as

$$0 = -\frac{\partial p}{\partial r} - \frac{\partial}{\partial z}\tau_{zr}$$

which is locally (in *r*) the same as the slit flow equation

Distributor Design (Power-Law)



- For flow down the circular tube

$$Q = Q_0 \left[1 - \frac{z}{L} \right]$$

- Equate this to the power-law result for $Q(dp/dz)$ for tube flow

$$Q_0\left[1-\frac{z}{L}\right]=\frac{\pi R^3}{\frac{1}{n}+3}\left(\frac{R}{2m}\right)^{\frac{1}{n}}\left(-\frac{dp}{dz}\right)^{\frac{1}{n}}$$

- Integrate to get the pressure drop down the tube

$$p-p_a=\frac{2mL}{R}\left(\frac{\frac{1}{n}+3}{\pi R^3}\right)^n\frac{Q_0^n}{n+1}\left(1-\frac{z}{L}\right)^{n+1}$$

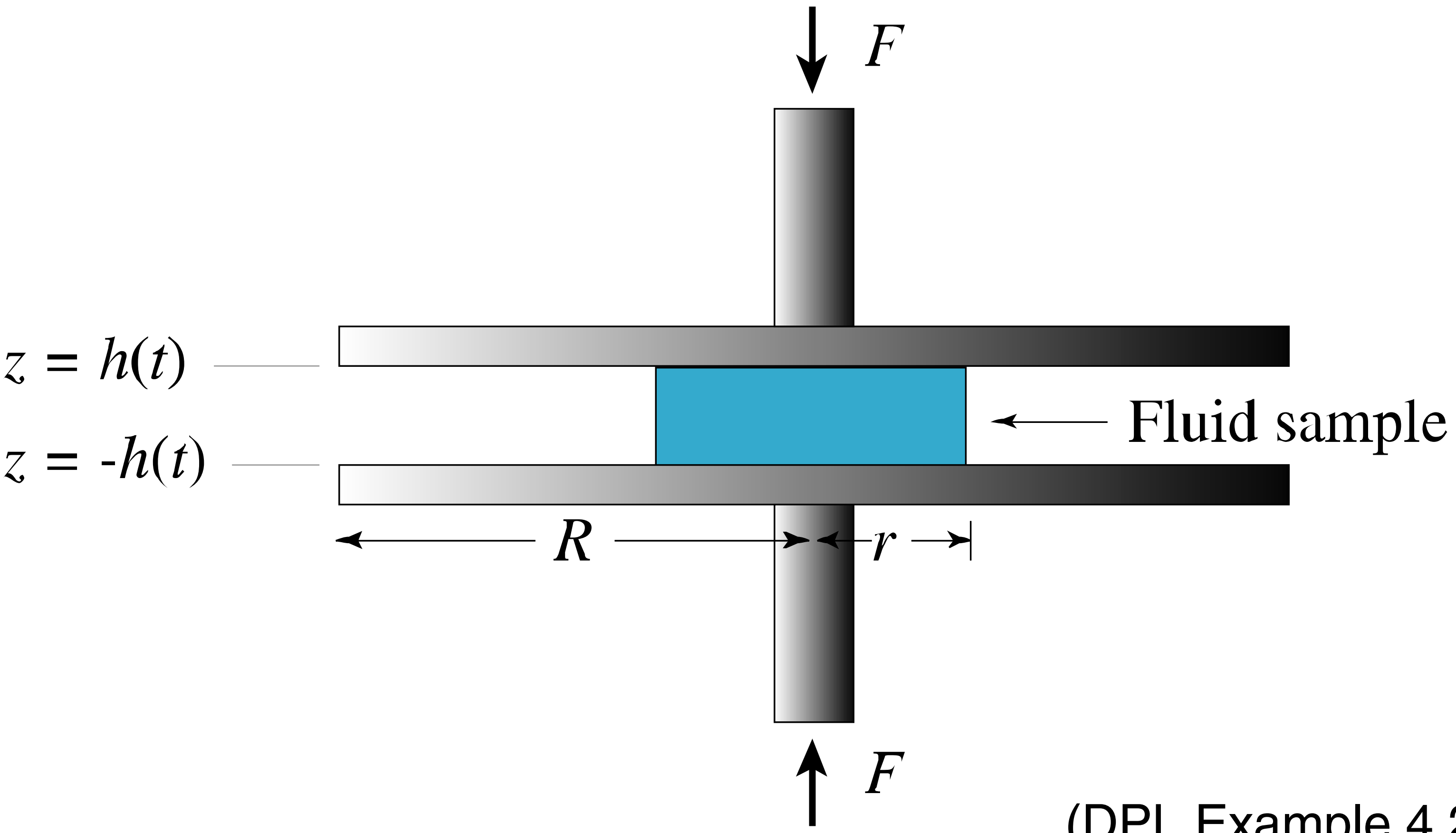
- At any position z, there is p - p_a driving force to force fluid through the slit of local length $l(z)$
- Slit flow for a power law fluid gives

$$V=\frac{B/2}{(1/n)+2}\left[\frac{(p-p_a)B}{2ml(z)}\right]^{\frac{1}{n}}$$

- Equating the available to needed pressure gradient gives *l*(*z*)

$$l(z) = \frac{BL}{R(n+1)}\left(\frac{B}{2\pi R^3}\frac{\left(1/n\right)+3}{\left(1/n\right)+2}\right)^n\left(\frac{Q_0}{V}\right)^n\left(1-\frac{z}{L}\right)^{n+1}$$

Squeezing Flow between Parallel Disks



(DPL Example 4.2-7)

- Volume flow rate across surface at *r*
  - Mass conservation

$$Q(r) = 2\pi r^2 \left(-\dot{h}\right)$$

  - Equation of motion with the lubrication approximation

Slit flow	Radial flow
W	$2\pi r$
B	h
$(P_0 - P_L)/L$	$-dp/dr$
Q	$Q(r)$

  - Hence

$$Q(r) = \frac{2 \cdot 2\pi r \cdot h^2}{(1/n)+2}\left(-\frac{h}{m}\frac{dp}{dr}\right)^{1/n}$$

- Solve for $p(r)$ with the boundary condition that $p(R) = p_a$

$$p - p_a = \frac{m\left(-\dot{h}\right)^n}{h^{2n+1}}\left(\frac{2n+1}{2n}\right)^n \frac{R^{n+1}}{n+1}\left[1-\left(\frac{r}{R}\right)^{n+1}\right]$$

- A force balance on the top plate gives

$$F(t) = \int_0^{2\pi}\int_0^{R}\left(p - p_a + \tau_{zz}\right)_{z=h} r\,dr\,d\theta$$

τ_{zz} = 0 on any solid surface

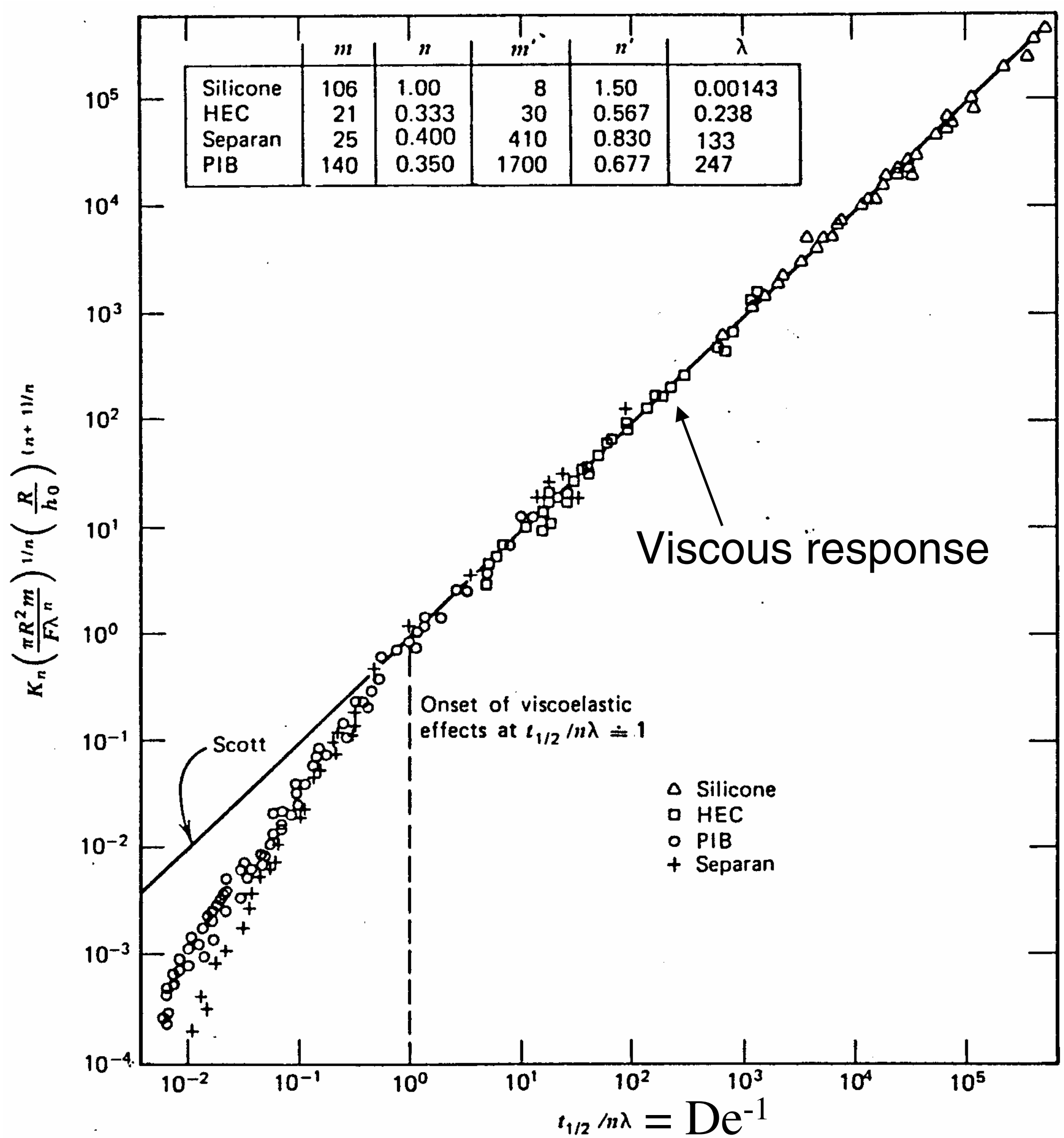
$$F = \frac{\left(-\dot{h}\right)^n}{h^{2n+1}}\left(\frac{2n+1}{2n}\right)^n \frac{\pi m R^{n+3}}{n+3}$$

Scott equation

- For constant force this can be integrated to give the half time $t_{1/2}$ for h to go from h_0 to $(½)h_0$:

$$\frac{t_{1/2}}{n} = K_n \left(\frac{\pi R^2 m}{F} \right)^{1/n} \left(\frac{R}{h_0} \right)^{1+(1/n)}$$

↑ (K_n): function of n



$$\lambda = (m'/2m)^{1/(n'-n)}$$

where

$$\eta = m\dot{\gamma}^{n-}$$

$$\Psi_1 = \ '\dot{\gamma}\ '-$$

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&RPPRQV OLFHQVH)RU PRUH LQIRUPDWLRQ VHH KWWSV RFZ PLW HGX KHOS IDT IDLU XVH