Lecture #7: Background

In section 5.5.2 (heralded by Fig 5.18), Frisk heads for the continuous modes of the Pekeris waveguide. The form for p(r,z) is shown explicitly as an integral in Eq. (5.175). At this point, we will leave Frisk and evaluate an equivalent, but slightly different, integral for the continuum following a paper by Tindle, Stamp, and Guthrie (TSG). A set of written notes exists for this. While the end results are the same, Frisk's treatment is somewhat more mathematically elegant, whereas Tindle's is more physically transparent – and both are worth reading.

Before getting into details of the Tindle et al. paper, let me digress again slightly on the quantum physics/ocean acoustics connection. Chris Tindle, the first author, was trained as a nuclear physicist (like me), and wandered a field into ocean acoustics in the 1970's as did I, as it was a booming field with lots of jobs then. This meant he had the physics "bag of tricks" at his disposal, and one of them, box normalization, is what he used in this paper. Cross discipline transfer of knowledge/techniques is a very useful thing, and also a lot of fun (as you get to dabble in two or more fields, and see a lot more things!)

Tindle's paper uses a rather different notation than Frisk's, but at a graduate level, that's just part of professional life and reading journal papers. 'Nuff said.

TSG starts at ground zero with the Helmholtz equation in cylindrical coordinates, which is OK as you can get a feel for the notation from this. One immediately sees that the N_n normalization diverges for the continuous modes, where the bottom half space is infinite. Addressing this, along with evaluating the continuum integral, are the major hurdles to jump. In the first part of the treatment, TSG define Z(z) in the bottom for the continuum, and then show how one can use the water-sediment interface BC's to get the amplitude A. The phase δ of the Z(z) is next. To get the surface BC right (treating the water as a thin layer, so we can go from zero to infinity with the continuous modes), we must set δ = 0, so we're ok there. Now, we just have to deal with the extra L we're introduced by the "box normalization." This is done in two steps. First, we use the normalization integral, which gives the trapped modes and a huge number of discrete continuous modes (and is still a proper S-L system!). The L still appears in the normalization at this point. The next trick, which gets rid of the L, is to change the continuous mode sum to an integral in the limit $L \rightarrow \infty$. This conversion gives a 1/L that cancels the L we introduced before! So in the limit, all is well!

At this point we have an integral free of infinite and arbitrary L factors – so let's evaluate it! This is a rather famous integral in ocean acoustics and seismology, and its exact evaluation is actually quite complicated. But TSG show a very nice, physical approximate evaluation, and it is a pretty good one for most purposes.

Since γ_2 (the vertical wave number in the bottom) is the integration variable and ranges from $0 \rightarrow \infty$, we can look at "regimes" of γ_2. It is seen that small γ_2 is the important part, and that the integral is dominated by the behavior of the denominator. In particular, the denominator is resonant where $\zeta'(H)=0$, which is just the condition for the hard bottom eigenvalue equation that we discussed previously! This seems odd, but on a moments reflection, we recall that the

phase of the reflection coefficient is zero for $\theta > \theta_{crit}$, so that each partial reflection of the sound off the bottom back into the waveguide had zero phase – thus the hard bottom condition!

Given a set of "continuum resonances" at the hard bottom eigenvalues, we can merrily evaluate these dominant contributions. TSG do this, and one sees as a result a set of "virtual modes" at the hard bottom eigenvalues that continue the set of the discrete Pekeris trapped modes! As we will see, these "virtuals" are very real modes, in fact.

Physically, the continuum virtual modes have an $e^{-\alpha_n r}$ behavior, where the α_n depends on the bottom density and sound speed. This behavior can actually be predicted very simply by looking at an "equivalent ray" reflection picture, with the plane wave reflection coefficient added in. (This is assigned as a homework problem).

As a final note on the continuum, it is found in practice that it dies down at a relatively small distance from the source for the Pekeris waveguide. So why care about it? The answer is scattering, which repopulates the continuum via mode coupling. The continuum is a "loss mechanism" that gets invoked very commonly in ocean acoustics!

There is yet another way to obtain the normal modes of the ocean waveguide, but a bit more roundabout. This is wavenumber integration (the Hankel transform approach). Its rationale is simple – the modes, as we've seen, are created by constructive interferences of up and down-going place waves. But at a point source in a waveguide (which immediately suggests a Greens function) produces spherical waves, not plane. This means we need to decompose a spherical wave into a weighted sum of plane waves – which is the "wavenumber integration." When we carry out this program, we indeed generate the modal picture, as we will see.

We start by looking at the problem of a point source over an infinite reflective halfspace (the sea bottom or surface). This is our old "Lloyds mirror" setup, and is shown in Figure 4.13. Eq. (4.90) is the Helmholtz eq. for the system, which we 2-D Fourier transform to get the "depth dependent Greene function" found in 4.92. The 2-D Fourier transformer pair between p and g is shown in Eqs. 4.93 and 4.94. We now want a solution to Eq. 4.92, which we get by incorporating the reflecting boundary conditions and using the standard endpoint method for Green's functions. Doing this gives the g in Eqs. 4.98 and 4.99. The forms one sees in Eqs. 4.101 and 4.102, which are integrated over the horizontal wavenumbers, are in fact the plane waves decomposition of the point source incident and reflected fields. So, first task accomplished!

The discussion just following Eqs. 4.101 and 4.102 also emphasizes that the decomposition must include homogeneous and inhomogeneous (imaginary/decaying) plane waves.

If one assumes cylindrical symmetry for the ocean waveguide (usually a good start, except on slopes and other regions of strong range dependence), one can fold the 2-D Fourier transform into a 1-D Hankel transform, shown in Eqs. 4.106 to 4.111. Note that β is the angle to the source and α is the wave vector angle. Doing this greatly simplifies things calculationally. The Hankel transform pair, Eqs. 4.108 and 4.110 will be the key to many developments.

The next part of the notes show how to use the Hankel transform pair to both see the modal structure of the full waveguide (water column and bottom) and also, how to use it to set up a nice inverse problem for the ocean bottom properties! (One can use either g (k_r) or R (k_r) to get c(z), α(Z), ρ(Z), and we've pursued both!).

The course notes show how one uses the endpoint method and HT to get the wave function for the point source in a full waveguide, and are pretty explicit. The slightly messy looking Green function shown in Step 7 is the uber-result. The key features of this g (k_r, z, z_0) are: 1) it has poles at the real mode eigenvalues and 2) the R in it can accommodate any type of layered bottom (i.e., it is very general).

Let's look at this g (k_r, z, z_0) for our two "well understood" waveguides, the hard bottom and Pekeris. First we need the R_B, which is a trivial $R_B = 1$ for a hard bottom; we also show the plot for a typical Pekeris. If we look at g for these cases, we see that there are sharp peaks (poles) at the trapped mode values, as expected. One intriguing result is that a "less peaky, but substantial, virtual mode peak shows for the Pekeris waveguide at just the hard bottom value!! Gratifying! We also see a phase flip of π at pole positions (why? ☺). If we insert the source in a null of a mode, we see a very distinct zero (and π phase jump). Interestingly, this works for both real and virtual modes! Well, this is maybe not so surprising if we look at the form we generated for the virtual modes.

One of the rationales for getting g (k_r, z, z_0) was to try to use it to get at the bottom properties, and we do that next. Our notes show the sequence for this. As a historical note, it was initially thought that the reflection coefficient we would obtain would be the best way to get at the bottom properties. It turned out that the Green's Function modal peaks were in fact easier to use, as they are large, robust features! However, R_B is still in play even today, though it is harder to obtain – there is a large body of formal inverse theory based on R_B that also makes it attractive.

The slide "Shallow Water HT experiment" shows a typical measurement of p(r), which can also be thought of as a synthetic aperture array. If we make this "simple" measurement (actually, not so damn simple), we can do a finite aperture Hankel transform (hmmmm – watsdat?) and see what we get for g (k_r). The pressure field we measure in the geometry shown has two trapped modes and one virtual mode, whose fields are shown in the next slide.

If we compare the g (k_r) for the infinite aperture and a 1000m finite aperture, for the example of a hard bottom, we get the (predictably smeared out) plot shown as "hard bottom." If we extend the aperture to 5000m, the peaks look much nicer, though the phases aren't quite right. (The reason for the latter is more subtle – let's pass on that one for now!). Next we look at the g (k_r) for the Pekeris waveguide for 1000m aperture and the "theoretical case" (infinite aperture). Again, the trapped peaks look a bit ratty for finite aperture, but the virtual mode looks ok! This is because the virtual mode p(r) dies off quickly in r, and so we only need a short aperture to "capture it." And again, for the 5000m aperture, the trapped peaks look better. (There is a standard Fourier $\Delta k \sim \pi / L$ relation that tells how well we can resolve the peak with a finite aperture– very useful).

Next, we look at how well we retrieve the reflection coefficient from g (k_r). It turns out, both for the hard bottom and the Pekeris waveguides – not very well. This is because R_B has a very complicated phase structure in terms of the g (k_r)and phase errors are pernicious and unforgiving. So initially at least, R_B was abandoned for doing inverses in favor of utilizing the g (k_r) peaks.

As a parting look at HT techniques, we look at Chapter 6 of Frisk, sections 6.1, 6.2 and 6.3. These sections fill in the derivation of the $p(r)$ and g (k_r, z, z_0) discussed in the previous instructors notes. They also show that, using the asymptotic form of $H_0^{(1)}$, we can approximate the HT by a FT and then of course use an FFT –> computer speed! This latter topic will be examined further in the numerical methods section.

We now take another step towards describing a more realistic ocean waveguide, one that varies in range along a source receiver track. This is almost a complete description for the acoustics problem; it only neglects out-of-plane, fully 3-D acoustic propagation. (We will get to that later.) The range-dependent Helmholtz equation (COA Eq. 5.265) is not strictly separable into a $\Phi(r)\Psi(z)$ type solution, so one uses the trick of defining "local modes" and a partially separable solution, with factors $\Phi_n(r)$ and $\Psi_n(r, z)$ as shown in Eq. 5.266 of COA (2nd Edition), where the local modes are defined in Eq. 5.267. Substitution of this solution into the Helmholtz equation gives an interesting result - the modes are coupled along the range, and thus exchange energy! (See Eqs. 5.27-5.273). This makes sense, in that the range dependent refraction/reflections change the grazing angles of the energy with range, thus sending it to different modes. One interesting bit about Eqs. 5.271-5.275 is that they totally ignore coupling to the continuum. But it is easy to see that energy can refract/scatter above the local critical grazing angle, so already something is missing. (Q: Can you replace the continuum by a few virtual modes here? This could be useful if it worked. Hmmm.)

Coupled mode calculations are a bit messy, though doable, and the results of seeing how the modes couple with range are often very enlightening. However, one can generally get a coupled mode result (including the continuum!) by using parabolic equation methods and then projecting out the local modes. So, on a scale of 1-10, I'm probably not a big fan of doing coupled mode calculations!

However, one can also often appeal to the "Laziness can be a virtue" theorem, and get something good out of it. We can just throw away the coupling coefficients (if the range dependence is "weak," the condition for which is not too hard to show) and get what is hilariously misnamed the "adiabatic solution" in which each mode keeps its own energy, but varies in range due to having a local modal wavenumber $k_n(r)$. This solution is shown in Eq. (5.280), and is so simple that it also should be put in ones arsenal of "simple analytical solution" tools. In section 5.11.4, COA shows a nice example of the range independent, adiabatic and coupled mode solutions in Fig. 5.29. The differences are apparent and striking!

The examples shown in Fig. 5.24 of COA are nice worked examples for the reader, but to get a "hands-on" feel for coupled and adiabatic modes, it is always useful to generate a "toy example" and play with its workings. This is done nicely in Katznelson and Petnikov's "Shallow

Water Acoustics", section 3.2. Written in a charming Russian-English prose (with a love of long sentences with many phrases), K & P start with the same local mode equation as COA, Eq 3.13, and then propose a radial WKB solution whose coefficients $C_l^{+/-}(r)$ are TBD. By ignoring backscatter (the $C_l^-(r)$) and terms with second derivatives in r (which says you take the radial direction curvature of any waveguide feature as small), K&P reduce things to a system of coupled first order DE's in r, to go along with the local modes. This is much easier to work with. The (only for first order) coupling coefficient for the system is shown in Eq. 3.19, and the rather simple solution for the $C_l^+(r)$ is shown in Eq. 3.20. If one sets the coupling coefficient V_{ml} to zero in Eq. 3.20, the adiabatic solution ala COA is again obtained; Eq. 3.21.

But, the more interesting part now is the coupling! In Eq. 3.23-3.24, K&P explicitly show the criterion for keeping/discarding the coupling (which we mentioned was simple – it is!). This is important in practical usage.

With a simple coupled mode equation, the coupling coefficient, and a solution in hand, we can explore the characteristics of a coupled mode system easily if we have a simple set of modes. The simplest, as we've seen, is the hard bottom waveguide, and thus K&P used just this waveguide with a variable bottom topography as their example. Eq. (3.25) shows the local modes for the example, and Eq. (3.27) the resulting coupling coefficient. "Close coupling" and the adiabatic criterion are shown clearly and cleanly in Eqs. (3.27-3.31). This toy problem in K&P is perhaps one of the clearest pedagogical examples to be found.

To conclude this long-ish lecture, there is a bit of simple "supplemental material" for those who might need a little brushing up on ocean acoustic normal modes (From Clay and Medwin) and end point methods for obtaining the Greens Function (from Myint-U). In engineering programs, academic backgrounds are often diverse, and there are occasional gaps that need a little filling!

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Spring 2012