MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.710 Optics **Spring '09**

Solutions to Problem Set #7 Due Wednesday, Apr. 22, 2009

Problem 1: Zernicke phase mask For problem 1, general formulations for the 4–f system are presented here. As shown in Fig. A in problem 1, x, x'', and x' are the lateral coordinates at the input, Fourier, and output plane, respectively. The complex transparencies at the input and Fourier plane are denoted by $t_1(x)$ and $t_2(x'')$, respectively. With on–axis plane illumination, we can formulate as follows:

1. field immediately after T1: $t_1(x)$
2. field immediately before T2: $\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}$
3. field immediately after T2: $t_2(x'')\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}$
4. field at the image plane: $\mathfrak{F}\left[t_2(x'')\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}\right]_{x''\rightarrow\frac{x'}{\lambda f_2}}$

$$= \mathfrak{F}\left[t_2(x'')\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} \otimes \mathfrak{F}\left[\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x'}{\lambda f_1}}\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} = \mathfrak{F}\left[t_2(x'')\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} \otimes t_1\left(-\frac{f_2}{f_1}x'\right), \tag{1}$$

where we use $\mathfrak{F}\left[\mathfrak{F}\left[g(x)\right]\right] = g(-x)$. Note that the field at the image plane is a convolution of the scaled object field and the Fourier transform of the pupil function, where the FT of the pupil is the point spread function of the system.

Next, it is important to model correctly the transparencies of the gratings. For T_1, the phase delay caused by grooves is $\frac{2\pi}{\lambda}(n-1)d_1$, where d_1 is the height of the groove (1 μm), and the phase profile is shown in Fig. 1. Hence, the complex transparency of

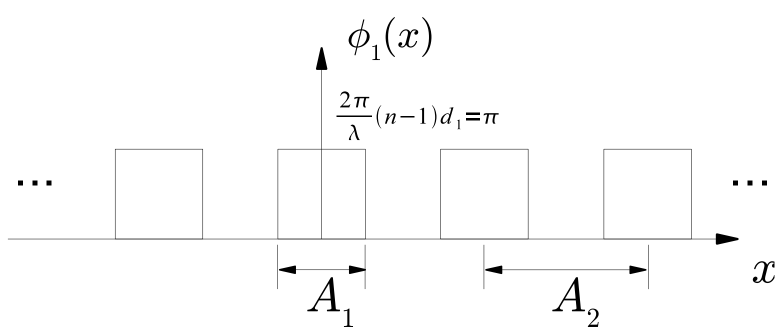


Figure 1: phase profile of the grating $T_1(x)$

T_1 is written as

$$t_1(x) = e^{i\phi_1(x)} = \exp\left\{ i\frac{2\pi}{\lambda}(n-1)d_1 \left[\text{rect}\left(\frac{x}{A_1}\right) \otimes \text{comb}\left(\frac{x}{A_2}\right) \right] \right\}, \tag{2}$$

where $A_1 = 5$ μm and $A_2 = 10$ μm. Hence,

$$t_1(x) = \begin{cases} e^{i\pi}(=-1) & \text{if } |x| < A_1/2, \\ 1 & \text{if } A_1/2 < x < A_2/2 \text{ or } -A_2/2 < x < -A_1/2, \end{cases} \tag{3}$$

for $|x| < A_2/2$. Using the Fourier series ($\because$ $t_1(x)$ is periodic) and $A = A_2 = 2A_1$, we find the Fourier series coefficients as

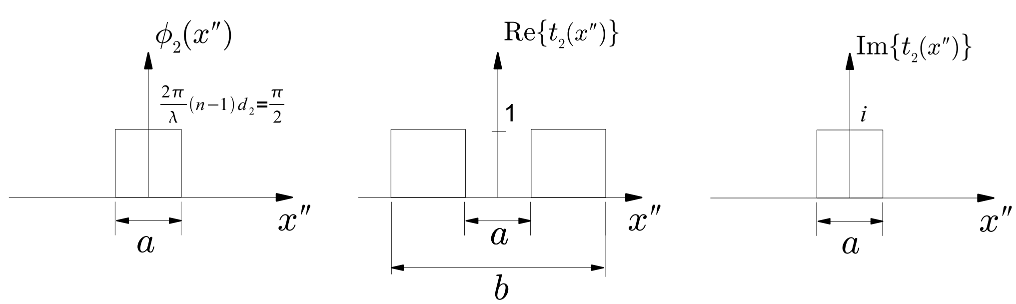
$$\begin{aligned} c_q &= \frac{1}{A}\int_{-A/2}^{A/2} t_1(x) e^{-i\frac{2\pi}{A}qx} \mathrm{d}x \\ &= \frac{1}{A}\left[\int_{-A/2}^{-A/4} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x - \int_{-A/4}^{A/4} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x + \int_{A/4}^{A/2} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x \right] \end{aligned} \tag{4}$$

For $q = 0$, $c_0 = \frac{1}{A}\int t_1(x)\mathrm{d}x = 0$.
For $q \neq 0$,

$$\begin{aligned} c_q &= \frac{1}{A}\left[\left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{-A/2}^{-A/4} - \left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{-A/4}^{A/4} + \left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{A/4}^{A/2} \right] \\ &= \frac{1}{A}\left[\frac{e^{i\frac{\pi}{2}q} - e^{i\pi q}}{-i\frac{2\pi}{A}q} - \frac{e^{-i\frac{\pi}{2}q} - e^{i\frac{\pi}{2}q}}{-i\frac{2\pi}{A}q} + \frac{e^{-i\pi q} - e^{-i\frac{\pi}{2}q}}{-i\frac{2\pi}{A}q} \right] \\ &= \frac{e^{i\pi q} - e^{-i\pi q}}{i2\pi q} - \left[\frac{e^{i\frac{\pi}{2}q} - e^{-i\frac{\pi}{2}q}}{i2\frac{\pi}{2}q} \right] = \text{sinc}\,(q) - \text{sinc}\left(\frac{q}{2}\right) = -\text{sinc}\left(\frac{q}{2}\right). \end{aligned}$$

Thus, $c_q = -\text{sinc}\left(\frac{q}{2}\right) + \delta(q)$; all even orders disappear and only odd orders survive.

For the grating T_2, the phase profile is shown in Fig. 2.



(a) phase profile (b) real part (c) real part

Figure 2: complex transparency of the grating $T_2(x'')$

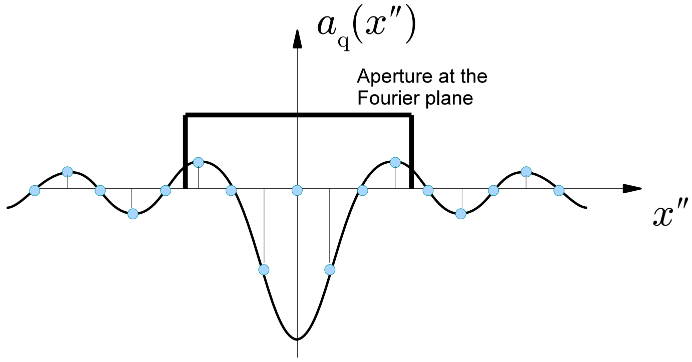


Figure 3: the field immediately before T_2.

The complex transparency can be written as

$$t_2(x'') = \left\{ \mathrm{rect}\left(\frac{x''}{b}\right) - \mathrm{rect}\left(\frac{x''}{a}\right) \right\} + i\,\mathrm{rect}\left(\frac{x''}{a}\right). \tag{5}$$

1.a) the intensity immediately after T1 is 1 because $|t_1(x)|^2 = 1$. Since T1 is a pure phase object and there is no intensity variation.

1.b) the field immediately before T2 can be computed from the Fourier series coefficients of $t_1(x)$. Since the period of T_1 is A, the diffraction angle of the order q is $\theta_q = q\frac{\lambda}{A}$, and the diffraction order q is focused at $f_1\theta_q$ on the Fourier plane. Hence, the field immediately before T_2 is

$$\sum_{q=-\infty}^{\infty} \left(\delta(q) - \mathrm{sinc}\,(q/2)\right) \delta\left(x'' - q\frac{f_1\lambda}{A}\right) = \sum_{q=-\infty}^{\infty} \left(\delta(q) - \mathrm{sinc}\,(q/2)\right) \delta\left(x'' - q \text{ cm}\right). \tag{6}$$

1.c) Since b (the width of the grating T_2) is 7 cm, the diffraction orders passing through the grating T_2 are $q = -3, -1, +1, +3$, where -1 and $+1$ orders get phase delay of $\pi/2$. The field immediately after the grating is

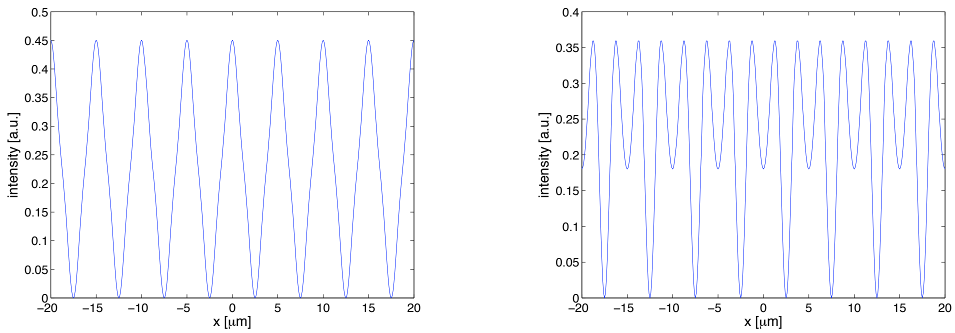
$$-\mathrm{sinc}\left(\frac{3}{2}\right)\left[\delta(x''-3) + \delta(x''+3)\right] - e^{i\frac{\pi}{2}}\mathrm{sinc}\left(\frac{1}{2}\right)\left[\delta(x''-1) + \delta(x''+1)\right]. \tag{7}$$

The field at the image plane is the Fourier transform of the field immediately after the grating T_2, which is computed as

$$\begin{aligned}
&\mathfrak{F}\left[-\mathrm{sinc}\left(\frac{3}{2}\right)\left[\delta(x''-3) + \delta(x''+3)\right] - i\mathrm{sinc}\left(\frac{1}{2}\right)\left[\delta(x''-1) + \delta(x''+1)\right]\right]_{u'' \to \frac{x'}{\lambda f}} = \\
&- 2\mathrm{sinc}\left(\frac{3}{2}\right)\mathfrak{F}\left[\frac{\delta(x''-q) + \delta(x''+q)}{2}\right] - 2i\mathrm{sinc}\left(\frac{1}{2}\right)\mathfrak{F}\left[\frac{\delta(x''-1) + \delta(x''+1)}{2}\right] = \\
&-2\mathrm{sinc}\left(\frac{3}{2}\right)\cos\left(2\pi 3u\right) - 2i\mathrm{sinc}\left(\frac{1}{2}\right)\cos\left(2\pi u\right) = -2\frac{-2}{3\pi}\cos\left(2\pi\frac{3x}{\lambda f_2}\right) - 2i\frac{2}{\pi}\cos\left(2\pi\frac{x}{\lambda f_2}\right) = \\
&\frac{4}{3\pi}\cos\left(\frac{2\pi}{\lambda}(0.3)x\right) - i\frac{4}{\pi}\cos\left(\frac{2\pi}{\lambda}(0.1)x\right).
\end{aligned} \tag{8}$$

The intensity at the image plane is

$$I(x) \sim \left| \frac{1}{3}\cos\left(\frac{2\pi}{\lambda}(0.3)x\right) - i\cos\left(\frac{2\pi}{\lambda}(0.1)x\right) \right|^2 = \frac{1}{9}\cos^2\left(\frac{2\pi}{\lambda}(0.3)x\right) + \cos^2\left(\frac{2\pi}{\lambda}(0.1)x\right) \quad (9)$$



(a) with the phase mask

(b) with the phase mask

Figure 4: intensity pattern at the image plane

Figure 4 shows the intensity pattern at the image plane with and without the phase mask.

1.d) In Fig. 4, the phase mask introduces more dramatic intensity contrast, whose frequency is proportional to the twice of the spatial frequency of the object grating. In Fig. 4(b), there is a intensity variation but the contrast is smaller. This phase mask is particularly useful for imaging phase object because phase variation is converted into intensity variation.

1.e) In Fig. 4(a), although all the orders are recovered, the field signal is not identical as the input field (the field immediately after T1). Hence, we may still able to observe some intensity variation although the contrast could be very limited (but still better than the case without the phase mask).

1.f) If $a = 0.5$ cm, then the first order does not get the phase delay, and all the orders are imaged at the image plane. The output field is identical to the input field (the field immediately after T1); no intensity variation is produced. Intuitively, in Fig. 4(b), as all the order contribute, the valleys of the intensity pattern is filled and eventually uniform intensity pattern is produced.

1. Another solution with an alternative definition of the grating T1

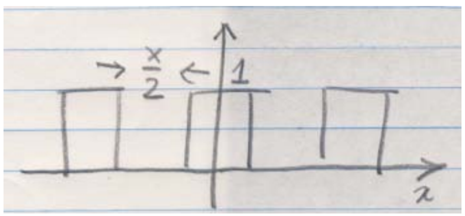
   (a) T1 is a phase mask; therefore, under the scalar and paraxial approximations, T1 does not modify the incident intensity.

$$I(x)\Big|_{\substack{\text{just}\\\text{after}\\\text{T1}}} = |e^{i\phi(x)}|^2 = 1 \qquad \text{(uniform)}$$

where $\underbrace{\phi(x)}_{\text{one period}} = \begin{cases} 0 & \text{if } |x| < 2.5\mu\text{m} \\ \frac{2\pi}{\lambda}(n-1)t = \frac{2\pi}{1\mu\text{m}}(1.5-1)\times 1\mu\text{m} = \pi & \text{if } 2.5\mu\text{m} < |x| < 5\mu\text{m} \end{cases}$

(periodically repeating with period = 10μm)

   (b) Just before T2, the optical field is the Fourier transform of T1, scaled by λf_1. Using the formula [Goodman p.126, Figure 5.21]



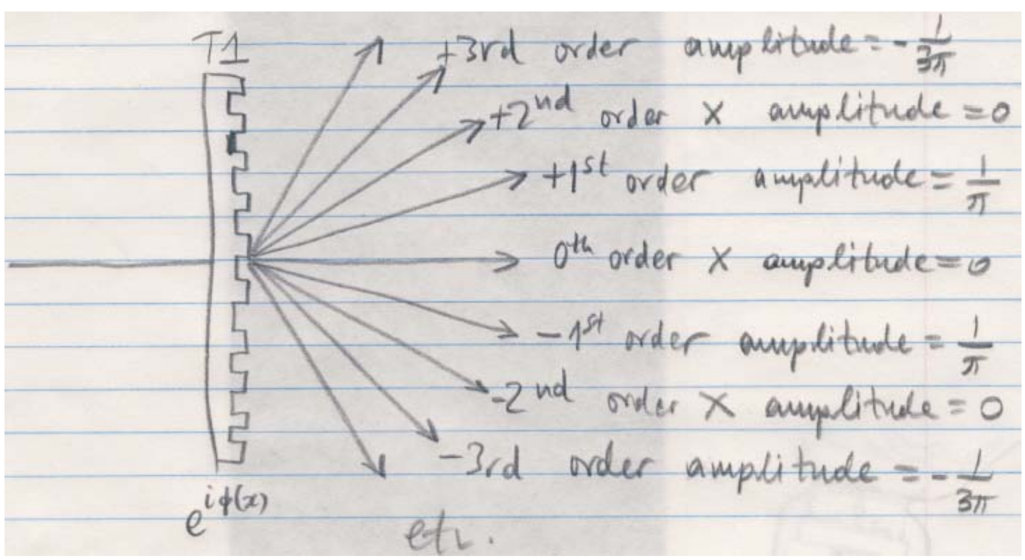
$$= \sum_{n=-\infty}^{\infty} \frac{\sin(\frac{\pi n}{2})}{\pi n} e^{i2\pi nx/x} \equiv a(x)$$

We obtain $e^{i\phi(x)} = 2[a(2) - \frac{1}{2}] = \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} \frac{\sin(\frac{\pi n}{2})}{\pi n} e^{i2\pi nx/x}$, where $x = 10\mu$m.

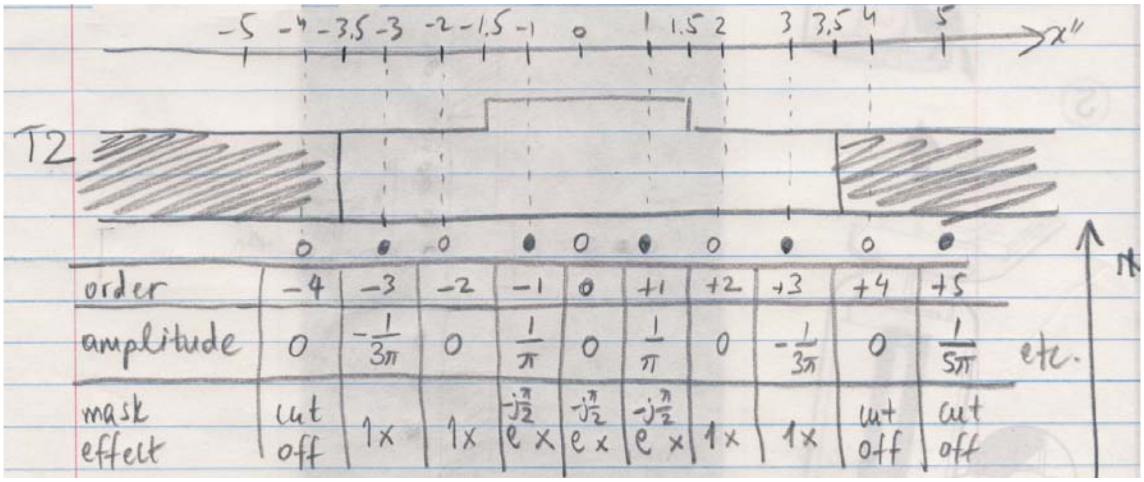
Note $\frac{\sin(\frac{n\pi}{2})}{n\pi}$ values:

n	$\ldots -5$	-4	-3	-2	-1	0	1	2	$3\ldots$
$\frac{\sin(n\pi/2)}{n\pi}$	$+\frac{1}{5\pi}$	0	$-\frac{1}{3\pi}$	0	$+\frac{1}{\pi}$	1	$+\frac{1}{\pi}$	0	$+\frac{1}{3\pi}$

These are the amplitudes of the diffracted orders.



Lens L1 focuses each diffracted order to a focal point just before T2:



The spacing between diffracted orders is:

$$\frac{\lambda f_1}{\text{grating period}} = \frac{1\mu\text{m} \times 10\text{cm}}{10\mu\text{m}} = 1\text{cm}$$

The phase shift applied by the elevated portion (center) of the T2 (pupil plane) mask is:

$$\frac{2\pi}{\lambda}(n-1)t_{T2} = \frac{2\pi}{1\mu\text{m}}(1.5-1) \times 0.5\mu\text{m} = \frac{\pi}{2}$$

So, <u>before</u> mask T2 the field is:

$\frac{1}{2}\left\{\cdots \frac{1}{5\pi}\delta(x''+5) - \frac{1}{3\pi}\delta(x''+3) + \frac{1}{\pi}\delta(x''+1) + \frac{1}{\pi}\delta(x''-1) - \frac{1}{3\pi}\delta(x''-3) + \frac{1}{5\pi}\delta(x''-5)\cdots\right\}$

(c) The mask T2 cuts off all orders beyond the $\pm 4^{\text{th}}$, and phase shifts the $\pm 1^{\text{th}}$ order with respect to the $\pm 3^{\text{rd}}$ orders. So just <u>after</u> mask T2 the field is:

$$\frac{1}{2}\left\{-\frac{1}{3\pi}\delta(x''+3) + \frac{e^{-i\frac{\pi}{2}}}{\pi}\delta(x''+1) + \frac{e^{-i\frac{\pi}{2}}}{\pi}\delta(x''-1) + \left(-\frac{1}{3\pi}\right)\delta(x''+3)\right\}$$

At the output plane, the field is the Fourier transform scaled by $\frac{1}{\lambda f_2}$ of the field just after T2, i.e. [recall $e^{-i\frac{\pi}{2}} = -i$]

$$\frac{1}{2}\left\{-\frac{1}{3\pi}e^{-i2\pi\frac{3x'}{10\mu\text{m}}} - \frac{i}{\pi}e^{-i2\pi\frac{x'}{10\mu\text{m}}} - \frac{i}{\pi}e^{+i2\pi\frac{x'}{10\mu\text{m}}} - \frac{1}{3\pi}e^{+i2\pi\frac{3x'}{10\mu\text{m}}}\right\}$$

$$= -\frac{1}{3\pi}\cos\left(2\pi\frac{3x'}{10\mu\text{m}}\right) - \frac{i}{\pi}\cos\left(2\pi\frac{x'}{10\mu\text{m}}\right)$$

The intensity is:

$$I_{\text{out}}(x') = \left|\text{field}_{\text{out}}(x')\right|^2 = \frac{1}{3\pi^2}\cos^2\left(2\pi\frac{3x'}{10\mu\text{m}}\right) + \frac{1}{\pi^2}\cos^2\left(2\pi\frac{x'}{10\mu\text{m}}\right)$$

(d) We recognize this as a Zernike phase mask. The original phase object T1 would have been invisible without the mask T2. With T2, we can observe intensity variation in the output plane.

Problem 2: Signum phase mask

2.a) The phase shift induced by the mask is

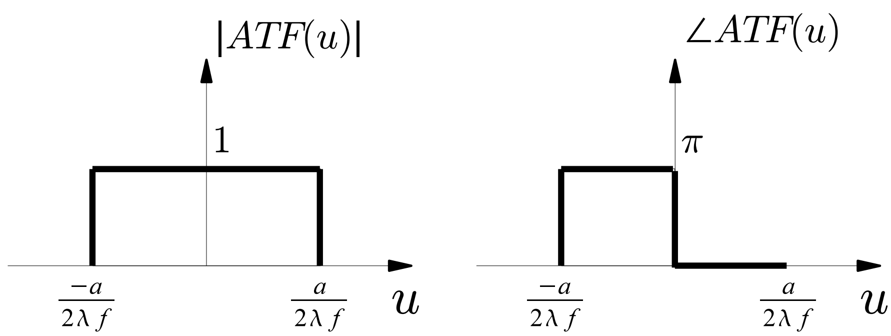
$$\frac{2\pi}{\lambda}s(n-1) = \frac{2\pi}{1\ \mu\text{m}}(1\ \mu\text{m})(0.5) = \pi. \tag{10}$$

Hence, the pupil function can be written as

$$P(x'') = \text{rect}\left(\frac{x''-a/4}{a/2}\right) + \text{rect}\left(\frac{x''+a/4}{a/2}\right)e^{i\pi}. \tag{11}$$

ATF is a scaled pupil function, which is found to be

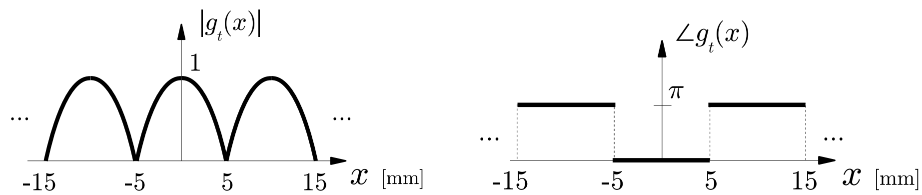
$$H(u) = P(\lambda f u) = \text{rect}\left(\frac{\lambda f u - a/4}{a/2}\right) + \text{rect}\left(\frac{\lambda f u + a/4}{a/2}\right)e^{i\pi}. \tag{12}$$



(a) magnitude (b) phase

Figure 5: ATF of the system

Note that $a/(2\lambda f) = \frac{1}{20}\ \mu\text{m}^{-1}$.
2.b) The magnitude and phase of the transparency are shown as below.



(a) magnitude (b) phase

Figure 6: Transparency

2.c) The optical field at the image plane can be obtained from two ways: 1) direct forward computation or 2) frequency analysis.

[1] direct forward computation: The incident field to the Fourier plane is

$$\mathfrak{F}\left[\cos\left(2\pi\frac{x}{\Lambda}\right)\right]_{\frac{x''}{\lambda f}} = \frac{1}{2}\delta\left(\frac{x''}{\lambda f}-\frac{1}{\Lambda}\right)+\frac{1}{2}\delta\left(\frac{x''}{\lambda f}+\frac{1}{\Lambda}\right) = \frac{1}{2}\delta(x''-5\text{ mm})+\frac{1}{2}\delta(x''+5\text{ mm}). \tag{13}$$

Since the width of the pupil is 10 mm, both delta functions pass through the pupil with a phase delay. The field immediately after the phase mask is

$$\frac{1}{2}\delta(x''-5)+e^{i\pi}\frac{1}{2}\delta(x''+5). = \frac{1}{2}\delta(x''-5)-\frac{1}{2}\delta(x''+5). \tag{14}$$

The field at the image plane is

$$\begin{aligned}\mathfrak{F}\left[\frac{1}{2}\delta(x''-5)-\frac{1}{2}\delta(x''+5)\right]_{\frac{x'}{\lambda f}} &= i\mathfrak{F}\left[\frac{1}{2i}\delta(x''-5)-\frac{1}{2i}\delta(x''+5)\right]_{\frac{x'}{\lambda f}} \\ &= i\sin\left(2\pi\frac{1}{5}\frac{x'}{\lambda f}\right) = i\sin\left(2\pi\frac{x'}{20\ \mu\text{m}}\right). \end{aligned} \tag{15}$$

[2] frequency analysis: The Fourier transform of the output field is a multiplication of the Fourier transform of the input field and the ATF. Since the Fourier transform of the input signal is $\frac{1}{2}\delta\left(u-\frac{1}{20\ \mu\text{m}}\right)+\frac{1}{2}\delta\left(u+\frac{1}{20\ \mu\text{m}}\right)$, the FT of the output field is

$$\frac{1}{2}\delta\left(u-\frac{1}{20\ \mu\text{m}}\right)-\frac{1}{2}\delta\left(u+\frac{1}{20\ \mu\text{m}}\right), \tag{16}$$

and the output field is $i\sin\left(2\pi\frac{x'}{20\ \mu\text{m}}\right)$.

2.d) Similarly, still there are two ways to analyze.

[1] If the incident wave is tilted, then one of the two delta functions at the Fourier plane is blocked by the pupil. The other delta function still gets phase delay and propagates to the image plane. The field immediately after the grating is

$$\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}\cos\left(2\pi\frac{x}{20\ \mu\text{m}}\right). \tag{17}$$

Then the field incident to the Fourier plane is

$$\begin{aligned}\mathfrak{F}\left[\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}\cos\left(2\pi\frac{x}{20\ \mu\text{m}}\right)\right]_{\frac{x''}{\lambda f}} &= \delta\left(\frac{x''}{\lambda f}-\frac{\theta}{\lambda}\right)\otimes\left\{\frac{1}{2}\delta\left(\frac{x''}{\lambda f}-\frac{1}{\Lambda}\right)+\frac{1}{2}\delta\left(\frac{x''}{\lambda f}+\frac{1}{\Lambda}\right)\right\} \\ &\frac{1}{2}\delta\left(x''-\theta f-\frac{\lambda f}{\Lambda}\right)+\frac{1}{2}\delta\left(x''-\theta f+\frac{\lambda f}{\Lambda}\right) \\ &= \frac{1}{2}\delta\left(x''-10\text{ mm}-5\text{ mm}\right)+\frac{1}{2}\delta\left(x''-10\text{ mm}+5\text{ mm}\right), \end{aligned} \tag{18}$$

where the second delta function does not get phase delay now. The field at the output plane is

$$\mathfrak{F}\left[\frac{1}{2}\delta\left(x''-5\text{ mm}\right)\right]_{\frac{x'}{\lambda f}} = \frac{1}{2}\exp\left\{-i2\pi\frac{x'}{\lambda f}(5\text{ mm})\right\} = \frac{1}{2}\exp\left\{i\frac{2\pi}{\lambda}(-0.05)x'\right\}. \tag{19}$$

thus, the output field is a tilted plane wave with an angle of -0.05 rad. The factor of $\frac{1}{2}$ indicates that the amplitude of the output field is half of the amplitude of the input field.

[2] The spatial frequency of the tilted plane wave is $u = \frac{\theta}{\lambda} = 0.1\ \mu\text{m}^{-1}$. Then, the field immediately after the grating is

$$\frac{1}{2}\delta\left(u - \frac{1}{10\ \mu\text{m}} - \frac{1}{20\ \mu\text{m}}\right) + \frac{1}{2}\delta\left(u - \frac{1}{10\ \mu\text{m}} + \frac{1}{20\ \mu\text{m}}\right). \tag{20}$$

Then, the FT of the output field is

$$\frac{1}{2}\delta\left(u - \frac{1}{20\ \mu\text{m}}\right), \tag{21}$$

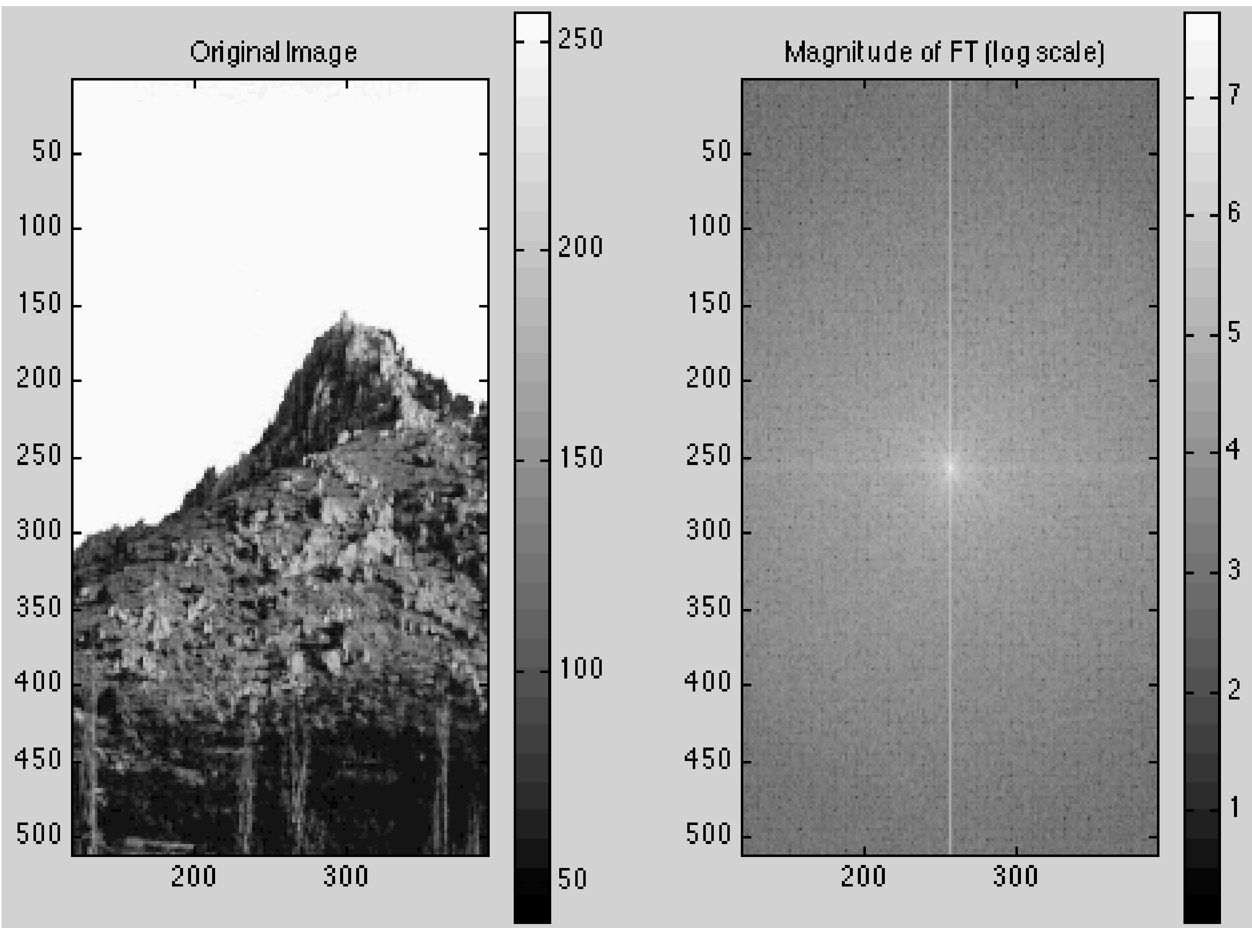
and the output field is

$$\frac{1}{2}\exp\left\{-i2\pi\left(\frac{1}{20}x'\right)\right\} = \frac{1}{2}\exp\left\{i\frac{2\pi}{\lambda}(-0.05)x'\right\}. \tag{22}$$

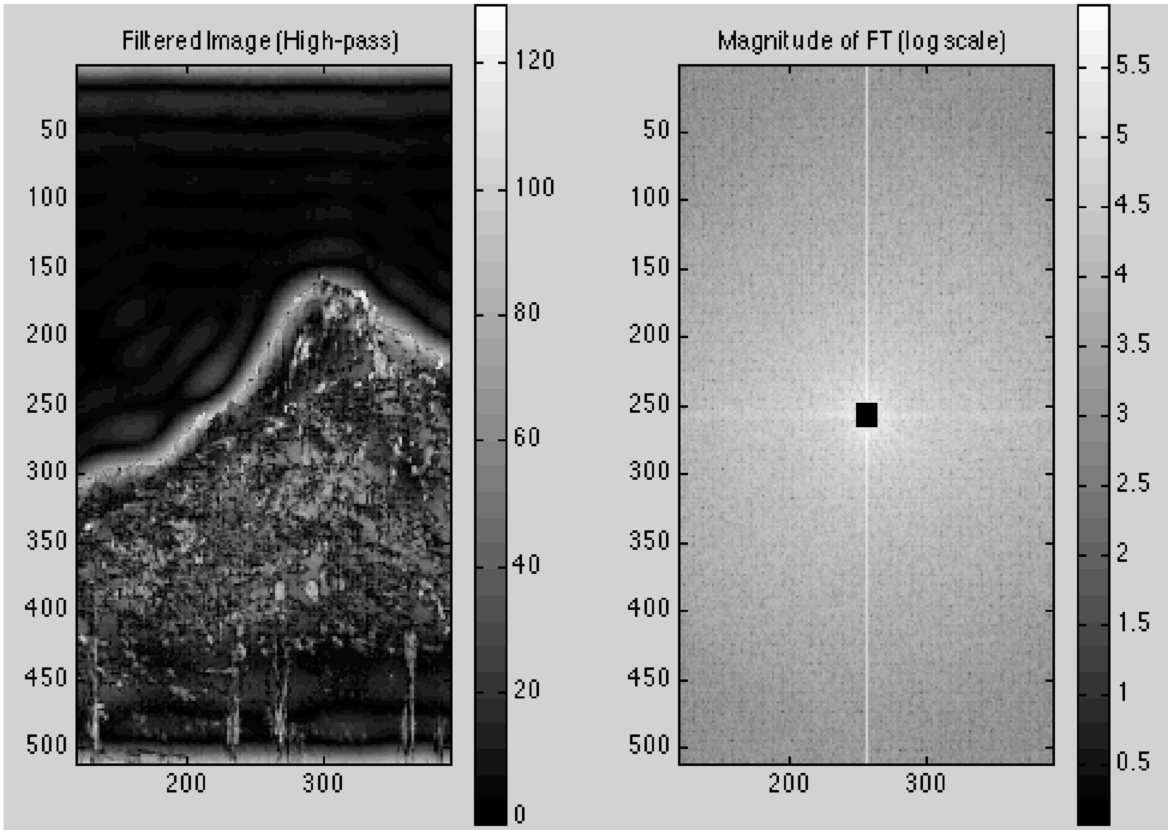
2.e) At the Fourier plane, a signum function with π phase shift is multiplied. It is equivalent to the Hilbert transform. (`http://en.wikipedia.org/wiki/Hilbert_transform`) Note that the Hilbert transform converts cos to sin and visa versa.

Problem 3

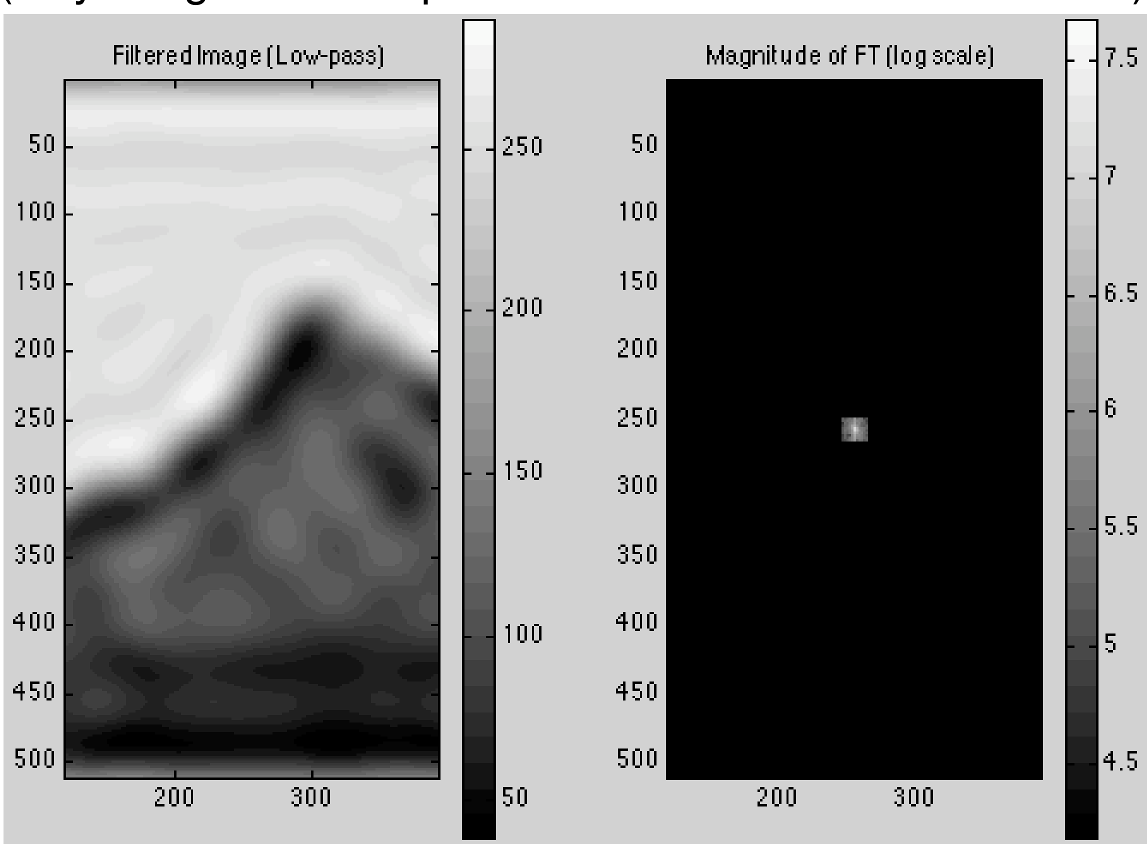
Plot of the Original Image and its Fourier Transform



High-pass Filter: Edges get enhanced



Low-pass Filter: Soften the Edges
(only the general shape of the mountain can be detected)



```
% Code for Problem 3, HW7, 2.710 (Optics)
% Solutions Fall 2001, Dario Gil

clear all;

x=8;%Pixels blocked = 2*(x+1)

A=imread('Tejera','jpg'); %Read the image to be processed


figure(1)

subplot(1,2,1), imagesc(A)
axis equal; colorbar;
title('Original Image')

subplot(1,2,2), imagesc(log10(abs(fftshift(fft2(A)))))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

% Part (b)
FTA=fftshift(fft2(A));
B=FTA;
i=length(A)/2-x:1:length(A)/2+(x+1);
j=i;
B(i,j)=0;

figure(2)
subplot(1,2,1), imagesc(abs(ifft2(B)))
axis equal; colorbar;

title('Filtered Image (High-pass)')
subplot(1,2,2), imagesc(log10(abs(B)))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

%part(c)

C=zeros(size(A));
C(i,j)=1;
C=C.*FTA;

figure(3)
subplot(1,2,1), imagesc(abs(ifft2(C)))
axis equal; colorbar;

title('Filtered Image (Low-pass)')
subplot(1,2,2), imagesc(log10(abs(C)))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

zoom on
```

Problem 4: Fourier transform

4.a) If the Fourier transform is linear, then it should satisfy two conditions

1. $\mathfrak{F}[f(x) + g(x)] = \mathfrak{F}[f(x)] + \mathfrak{F}[g(x)]$, where $f(x)$ and $g(x)$ are input functions to the Fourier transform

2. $\mathfrak{F}[af(x)] = a\mathfrak{F}[f(x)]$, where a is a constant.

For condition 1,

$$\mathfrak{F}[f(x) + g(x)] = \int \{f(x) + g(x)\} e^{-i2\pi xu} \mathrm{d}x = \int f(x) e^{-i2\pi xu} \mathrm{d}x + \int g(x) e^{-i2\pi xu} \mathrm{d}x = \mathfrak{F}[f(x)] + \mathfrak{F}[g(x)]. \quad (23)$$

For condition 2,

$$\mathfrak{F}[af(x)] = \int \{af(x)\} e^{-i2\pi xu} \mathrm{d}x = a \int f(x) e^{-i2\pi xu} \mathrm{d}x = a\mathfrak{F}[f(x)]. \quad (24)$$

Hence, the Fourier transform is a linear transform.

4.b) The transfer function is not defined in this case because the Fourier integral is not a form of convolution. In other words, the relation is not shift invariant.

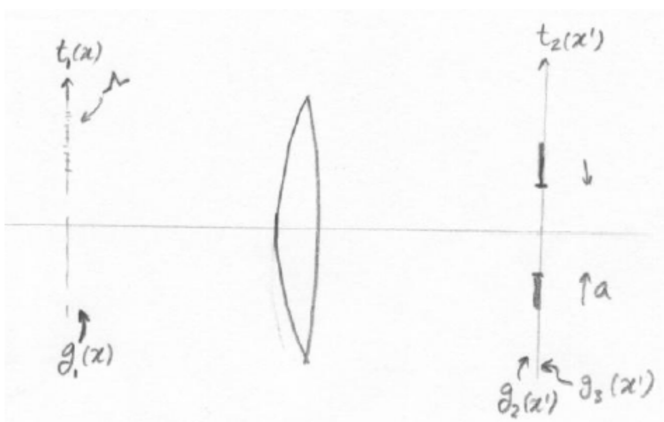
5.

$$\begin{aligned} g(x', y') &= f(x, y) \otimes h(x, y) \\ g(x') &= \mathcal{F}^{-1}\{\mathcal{F}\{g\}\} \\ G(u) &= \mathcal{F}\{g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{h\} \\ &= \left[\frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u - \frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u + \frac{1}{\Lambda}\right)\right] \cdot x_0 \text{rect}(x_0, u) \end{aligned}$$

$$G(u) = \begin{cases} \frac{1}{2}\delta(u) & \text{if } \frac{1}{2x_0} < \frac{1}{\Lambda} \\ \frac{1}{2}\delta(u) + \frac{1}{4}\delta(u - \frac{1}{\Lambda}) + \frac{1}{4}\delta(u + \frac{1}{\Lambda}) & \text{if } \frac{1}{2x_0} > \frac{1}{\Lambda} \end{cases}$$

$$g(x') = \begin{cases} \frac{1}{2}e^{i\phi} & \text{if } \frac{1}{2x_0} < \frac{1}{\Lambda} \text{ (plane wave)} \\ \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x}{\Lambda}\right)\right] & \text{if } \frac{1}{2x_0} > \frac{1}{\Lambda} \text{ (all the input gets imaged at the output)} \end{cases}$$

6. First create a labeled sketch of the problem.



(a)

$$\begin{aligned} g_1(x) &= t_1(x) = \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x}{\Lambda}\right)\right] \\ g_2(x') &= G_1(u)\Big|_{u=\frac{x'}{\lambda f}} = \frac{1}{2}\delta(x') + \frac{1}{4}\delta\left(x' - \frac{\lambda f}{\Lambda}\right) + \frac{1}{4}\delta\left(x' + \frac{\lambda f}{\Lambda}\right) \\ g_3(x') &= g_2(x') \cdot t_2(x') = \left[\frac{1}{2}\delta(x') + \frac{1}{4}\delta\left(x' - \frac{\lambda f}{\Lambda}\right) + \frac{1}{4}\delta\left(x' + \frac{\lambda f}{\Lambda}\right)\right] \cdot \text{rect}\left(\frac{x'}{a}\right) \\ &= \begin{cases} \frac{1}{2}\delta(x') & \text{if } \frac{a}{2} < \frac{\lambda f}{\Lambda} \\ \frac{1}{2}\delta(x') + \frac{1}{4}\delta\left(x' - \frac{\lambda f}{\Lambda}\right) + \frac{1}{4}\delta\left(x' + \frac{\lambda f}{\Lambda}\right) & \text{if } \frac{a}{2} > \frac{\lambda f}{\Lambda} \end{cases} \end{aligned}$$

(b) In this problem, we take the Fourier transform of $t_1(x)$ and multiply it with a rect() function. This is equivalent to taking the inverse Fourier of the rect() function, which is a sinc() function, convolving it with $t_1(x)$ and taking the Fourier of the convolution, which is what we did in problem # 5.