Outline:

- *Fresnel Diffraction*
- *The Depth of Focus and Depth of Field(DOF)*
- *Fresnel Zones and Zone Plates*
- *Holography*

A. Fresnel Diffraction

For the general diffraction problem, the electric field $E(x',y')$ measured at a distance z from the plane of the aperture is a convolution of three factors:

$$E(x', y') = \iint h(x' - x, y' - y, z)t(x, y)E(x, y)dxdy \tag{1}$$

$$h(x' - x, y' - y, z) = \frac{\exp(ikr)}{r} \tag{2}$$

where
$$r = \sqrt{(x' - x)^2 + (y' - y)^2 + z^2} \tag{3}$$

When the distance z is sufficiently far (z>>x',y', x, y) we take the paraxial approximation:

$$r \approx z\left(1 + \frac{(x'-x)^2 + (y'-y)^2}{2z^2}\right) \tag{4}$$

$$\exp(ikr) \approx \exp(ikz)exp\left(-ik\frac{xx'+yy'}{z}\right)\exp\left(ik\frac{x'^2+y'^2+x^2+y^2}{2z}\right) \tag{5}$$

It is the value of the quadratic term $k\frac{x'^2+y'^2+x^2+y^2}{2z}$ that determines whether the Fresnel or Fraunhofer approximation should be used. Generally speaking, it is determined according to whether the value of $k\frac{x'^2+y'^2+x^2+y^2}{2z}$ is larger than π/2 (Fresnel) or smaller than π/2 (Fraunhofer). For example, taking D=1mm, λ=500nm, then z(Fraunhoffer)=1m! Therefore between z=10mm to 1 m is all Fresnel diffraction region.

- **Fresnel propagator or Fresnel kernel:**

Two types of expressions for the Fresnel approximation can be obtained; one is in the form of a convolution and the other is in the form of a Fourier transform. If we expand exp(ikr) into quadratic terms:

$$E(x', y') = \iint h(x' - x, y' - y, z)t(x, y)E(x, y)dxdy \tag{6}$$

$$E(x', y') = \frac{\exp(ikz)}{z}\exp\left(ik\frac{x'^2+y'^2}{2z}\right)\iint exp\left(-ik\frac{xx' + yy'}{z}\right)\exp\left(ik\frac{x^2+y^2}{2z}\right)t(x, y)E(x, y)dxdy$$

$$\boxed{E(x', y') = \frac{\exp(ikz)}{z}\exp\left(ik\frac{x'^2+y'^2}{2z}\right)\mathcal{F}\left[\exp\left(ik\frac{x^2+y^2}{2z}\right)t(x, y)E(x, y)\right]} \tag{7}$$

You may recognize $\exp\left(ik \frac{x^2+y^2}{2z}\right)$ is a Gaussian function with respect to x and y, and the wavefront is diverging.

$$\text{Using } x' = k_x \frac{z}{k}, y' = k_y \frac{z}{k} \tag{8}$$

The corresponding transfer function is:

$$H(k_x, k_y) = \frac{\exp(ikz)}{z} \iint \exp\left(ik \frac{x^2+y^2}{2z}\right) \exp\left(-ik_x x - ik_y y\right) dx dy$$

$$H(k_x, k_y) = \frac{\exp(ikz)}{k} \exp\left(-iz \frac{k_x^2 + k_y^2}{2\pi k}\right) \tag{9}$$

The Fourier transform of a Gaussian function is still a Gaussian function. The above property is often used in analyzing the depth of focus (DOF).

B. The Depth of Focus (DOF)
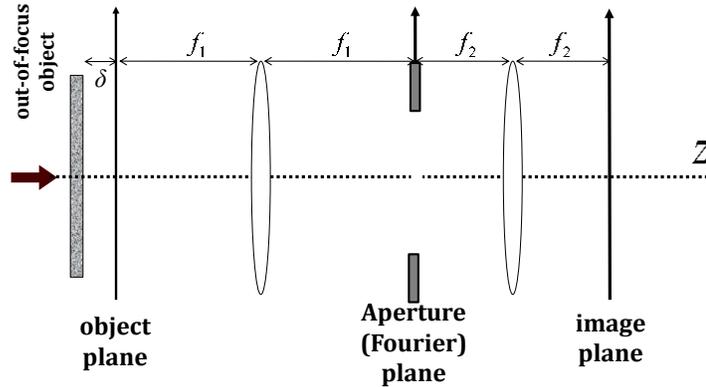
When a focusing error z=δ is present in the imaging system, there is a difference of path length from the "ideal" object plane. This means the field at the object plane is of the form:

$$E(x,y) \otimes \exp(ik\frac{x^2+y^2}{2\delta}) \qquad (10)$$

Correspondingly, the Fourier spectrum of the object is modified by $H(k_x,k_y)$:

$$E(k_x,k_y) \times \exp(-i\delta\frac{k_x{}^2+k_y{}^2}{2k}) \qquad (11)$$

Keep in mind, $k_x = k\frac{x'}{f_1}$, $k_y = k\frac{y'}{f_1}$. Therefore, the effect of defocus is like a phase mask, where the offset from the object plane create a quadratic phase shift for every **k** component on the aperture plane. Correspondingly, the out-of-focus point spread function is modified:

$$\text{PSF(defocus)} = \mathcal{F}\left[AS(k_x\frac{f_1}{k},\ k_y\frac{f_1}{k}) \times \exp(-i\delta\frac{k_x{}^2+k_y{}^2}{2k})\right] \qquad (12)$$

- **The significance of the defocus: (Goodman 6.4.4)**

Mild defocus: $\exp(-i\delta\frac{k_x{}^2+k_y{}^2}{2k}) \approx 1$ (13)

This requirement is met when

$$\delta\frac{k_x{}^2}{2k} = \frac{\delta}{2k}(k\frac{x'}{f_1})^2 \ll \frac{\pi}{2} \qquad (14)$$

Or

$$\frac{\delta k}{\pi}(NA)^2 \ll 1 \qquad (15)$$

$$\delta \ll \frac{\lambda}{2(NA)^2} \equiv DOF \text{ (Depth of Focus)} \qquad (16)$$
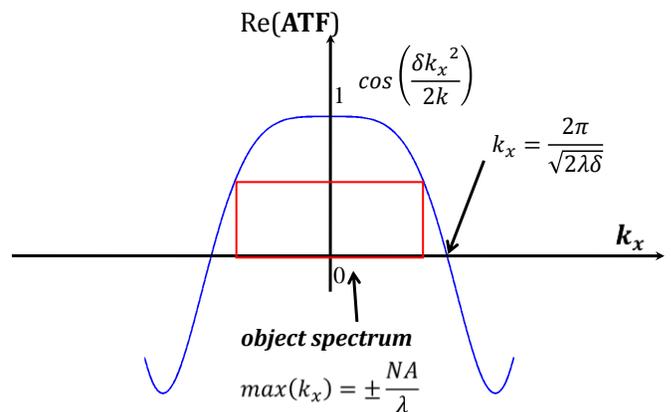
a.   Severe defocus: $\boxed{\delta \geq \dfrac{\lambda}{2(NA)^2}}$

In this case, the oscillatory nature of the defocus kernel results in strong blur on the image because of the suppression of spatial frequencies near the nulls and sign changes at the negative portions.
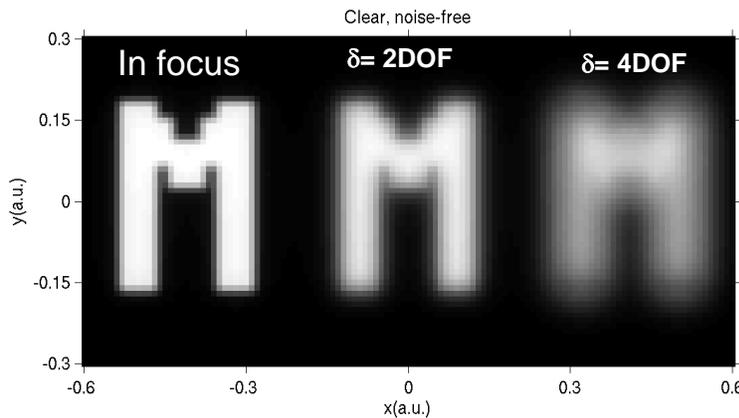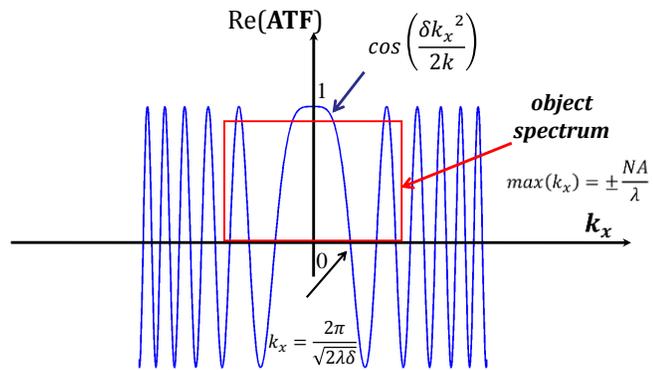
Re(**ATF**)

$cos\left(\dfrac{\delta k_x{}^2}{2k}\right)$

object spectrum

$max(k_x) = \pm\dfrac{NA}{\lambda}$

k_x

$k_x = \dfrac{2\pi}{\sqrt{2\lambda\delta}}$

Clear, noise-free

In focus δ= 2DOF δ= 4DOF

Figure: Computed imaging of letter "M" convolved with diffraction-limited PSF *at different degrees of* defocus.

- **Can the blur be undone computationally? (Goodman 8.8)**
  - **Inverse of Fresnel propagator** $H(k_x, k_y)$ over distance z:
    The problem of division is typically reduced to obtaining the transmittance of the inverse, namely:

    $$\frac{1}{H(k_x,k_y)} = \frac{H^*(k_x,k_y)}{\left|H(k_x,k_y)\right|^2} \tag{17}$$

    Note: this inverted filter is also limited by the numerical aperture; it may also include the effect of defocus and higher-order aberrations.

    In order to retrieve the proper information with noise, different statistical tools such as Tikhonov regularization are used.

  - *Practical limitations*: the inversion is sensitive to both noise in the measured data, and the accuracy of the assumed knowledge.
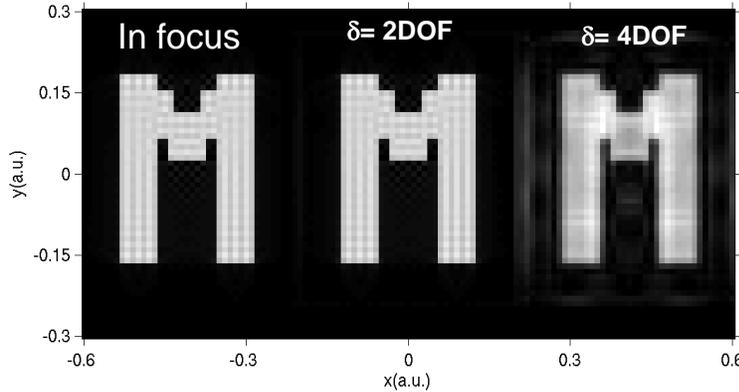
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Figure: Deconvolution using Tikhonov regularized inverse filter, utilized *a priori* knowledge of depth of each digit. Note the artifacts primarily due to numerical errors getting amplified by the inverse filter (despite regularization)

C. Fresnel Zones and Zone Plates

In the above analysis, we find that the shift of an object along the z axis is equivalent to a phase mask of varying phase delay in the aperture plane:

$$H(k_x, k_y) = \exp\left(-iz\frac{k_x^2 + k_y^2}{2k}\right) \qquad (18)$$

What happens if we placed an amplitude mask with the transmittance in the following form?

$$t(x, y) = \left[1 + \cos\left(\frac{x^2 + y^2}{2L}\right)\right] \qquad (19)$$

To answer this question we can calculate the Fresnel diffraction pattern of this system using $k_x = k\frac{x'}{z}, k_y = k\frac{y'}{z}$.

$$E(x', y') \approx \iint \exp\left(ik\frac{x^2 + y^2}{2z}\right)\left\{1 + \cos\left[\frac{x^2 + y^2}{2L}\right]\right\}\exp\{-i[k_x x + k_y y]\}dxdy$$

$$E(x', y') \approx \mathcal{F}\left\{\exp\left(ik\frac{x^2 + y^2}{2z}\right) + \frac{1}{2}\exp\left[ik(x^2 + y^2)\left(\frac{1}{2L} + \frac{1}{2z}\right)\right] + \frac{1}{2}\exp\left[-ik(x^2 + y^2)\left(\frac{1}{2L} - \frac{1}{2z}\right)\right]\right\} \qquad (20)$$

The Fourier transform of the first term is straight forward:

$$\exp\left(-ik\frac{x'^2 + y'^2}{2z}\right). \qquad (21)$$

Likewise, we can express the second and the third term:

$$\frac{1}{2}\exp\left(-ik\,\frac{L}{2z}\,\frac{x\prime^2+y\prime^2}{L+z}\right)+\frac{1}{2}\exp\left(ik\,\frac{L}{2z}\,\frac{x\prime^2+y\prime^2}{z-L}\right). \qquad (22)$$

the 3rd term indicates a converging wave front towards z=L (a real image) on the optical axis, while as the 2nd term indicates a diverging wave front from a source located at z=-L (a virtual image) behind the aperture.

This is known as a **Gabor zone plate** (the building block of a hologram). Such plates can be produced optically by photographing the interference pattern formed by two coherent spherical wavefront of different radii of curvature.
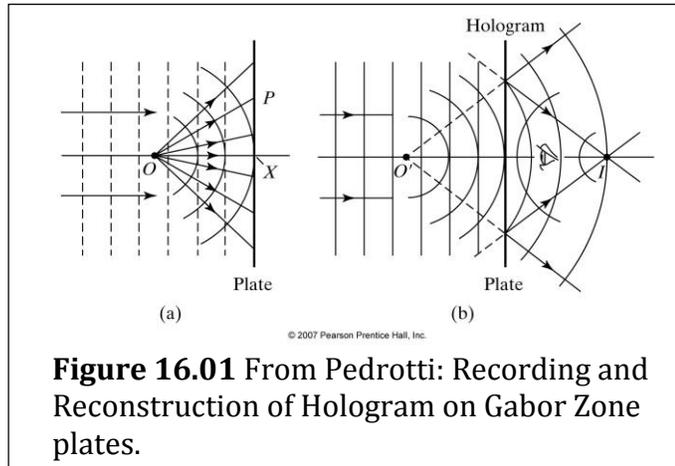
Figure 16.01 From Pedrotti: Recording and Reconstruction of Hologram on Gabor Zone plates.

More general idea of such plates in amplitude or phase can be constructed such that the phase differs by π from one boundary to the next. The *m*th boundary has radius determined by

$$\frac{k}{2}\left(x^2+y^2\right)\left(\frac{1}{z\prime}+\frac{1}{z}\right)=m\pi \qquad (23)$$

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