MIT 2.852
Manufacturing Systems Analysis
Lectures 6–9: Flow Lines

Models That Can Be Analyzed Exactly

Stanley B. Gershwin

http://web.mit.edu/manuf-sys

Massachusetts Institute of Technology

Spring, 2010

Copyright ©2010 Stanley B. Gershwin.

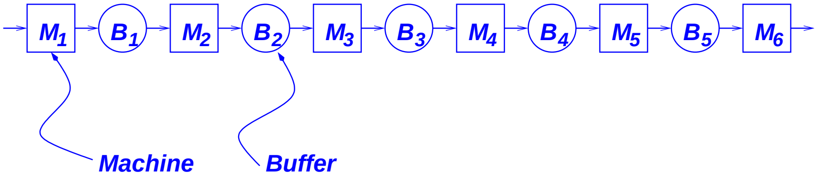
Models

- *The purpose of an engineering or scientific model is to make predictions.*
- Kinds of models:
  - *Mathematical:* aggregated behavior is described by equations. Predictions are made by solving the equations.
  - *Simulation:* detailed behavior is described. Predictions are made by reproducting behavior.
- Models are simplifications of reality.
  - Models that are too simple make poor predictions because they leave out important features.
  - Models that are too complex make poor predictions because they are difficult to analyze or are time-consuming to use, because they require more data, or because they have errors.

 Copyright ©2010 Stanley B. Gershwin.

Flow Line

... also known as a Production or Transfer Line.



- Machines are unreliable.
- Buffers are finite.

 Copyright ©2010 Stanley B. Gershwin.

- Economic importance.
- Relative simplicity for analysis and for intuition.

 Copyright ©2010 Stanley B. Gershwin.

Flow Line

Buffers and Inventory

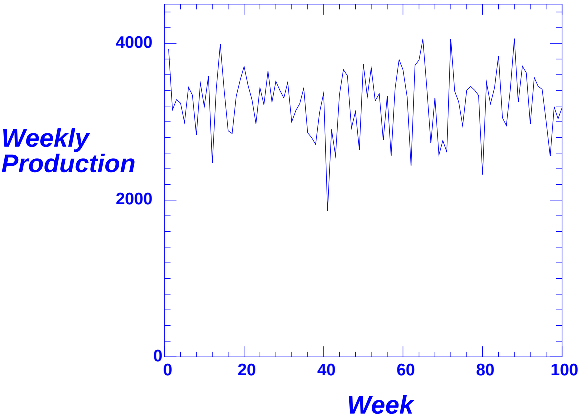
- Buffers are for mitigating asynchronization (ie, they are shock absorbers).
- Buffer space and inventory are expensive.

Copyright ©2010 Stanley B. Gershwin.

- Complex behavior.
- Analytical solution available only for limited systems.
- Exact numerical solution feasible only for systems with a small number of buffers.
- Simulation may be too slow for optimization.

 Copyright ©2010 Stanley B. Gershwin.

Flow Line
Output Variability



Production output from a simulation of a transfer line.

 Copyright ©2010 Stanley B. Gershwin.

Flow Line
Usual General Assumptions

- Unlimited repair personnel.
- Uncorrelated failures.
- Perfect yield.
- The first machine is never starved and the last is never blocked.
- Blocking before service.
- Operation dependent failures.

 Copyright ©2010 Stanley B. Gershwin.

Single Reliable Machine

- If the machine is perfectly reliable, and its average operation time is τ, then its maximum production rate is $1/\tau$.
- *Note:*
  - Sometimes *cycle time* is used instead of *operation time*, but *BEWARE:* cycle time has two meanings!
  - The other meaning is the time a part spends in a system. If the system is a single, reliable machine, the two meanings are the same.

 Copyright © 2010 Stanley B. Gershwin.

- Operation-Dependent Failures
  - A machine can only fail while it is working.
  - *IMPORTANT!* MTTF *must* be measured in working time!
  - This is the usual assumption.
- *Note:* MTBF = MTTF + MTTR

 Copyright ©2010 Stanley B. Gershwin.

Single Unreliable Machine
Production rate

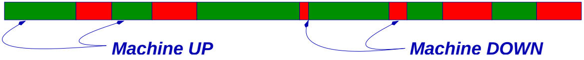
- If the machine is unreliable, and
  - its average operation time is τ,
  - its mean time to fail is MTTF,
  - its mean time to repair is MTTR,

  then its maximum production rate is

$$\frac{1}{\tau}\left(\frac{\text{MTTF}}{\text{MTTF}+\text{MTTR}}\right)$$

 Copyright ©2010 Stanley B. Gershwin.

Proof



- Average production rate, while machine is up, is $1/\tau$.
- Average duration of an up period is MTTF.
- Average production during an up period is MTTF$/\tau$.
- Average duration of up-down period: MTTF + MTTR.
- Average production during up-down period: MTTF$/\tau$.
- Therefore, average production rate is (MTTF$/\tau$)/(MTTF + MTTR).

 Copyright ©2010 Stanley B. Gershwin.

- *Assumptions:* Operation time is constant (τ). Failure and repair times are *geometrically* distributed.

- Let p be the probability that a machine fails during any given operation. Then $p = \tau/\text{MTTF}$.

 Copyright ©2010 Stanley B. Gershwin.

Single Unreliable Machine

- Let r be the probability that M gets repaired in during any operation time when it is down. Then $r = \tau/\text{MTTR}$.

- Then the *average production rate* of M is

$$\frac{1}{\tau}\left(\frac{r}{r+p}\right).$$

- *(Sometimes we forget to say "average.")*

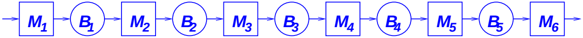
 Copyright ©2010 Stanley B. Gershwin.

Single Unreliable Machine
Production Rates

- So far, the machine really has *three* production rates:
  - $1/\tau$ when it is up *(short-term capacity)* ,
  - 0 when it is down *(short-term capacity)* ,
  - $(1/\tau)(r/(r+p))$ on the average *(long-term capacity)* .

Copyright ©2010 Stanley B. Gershwin.

Infinite-Buffer Line

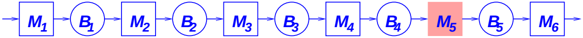


Assumptions:

- A machine is not idle if it is not starved.
- The first machine is never starved.

Copyright ©2010 Stanley B. Gershwin.

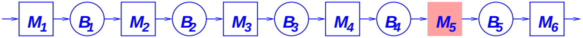
Infinite-Buffer Line



- The production rate of the line is the production rate of the *slowest* machine in the line — called the *bottleneck* .
- *Slowest* means least average production rate, where average production rate is calculated from one of the previous formulas.

 Copyright ©2010 Stanley B. Gershwin.

Infinite-Buffer Line



- Production rate is therefore

$$P = \min_i \frac{1}{\tau_i} \left(\frac{\text{MTTF}_i}{\text{MTTF}_i + \text{MTTR}_i} \right)$$

- and M_i is the bottleneck.

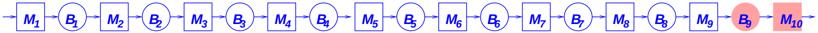
 Copyright ©2010 Stanley B. Gershwin.

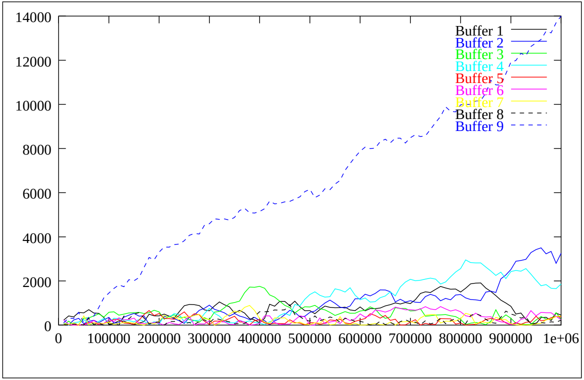
Infinite-Buffer Line

- The system is not in steady state.
- An infinite amount of inventory accumulates in the buffer upstream of the bottleneck.
- A finite amount of inventory appears downstream of the bottleneck.

 Copyright ©2010 Stanley B. Gershwin.

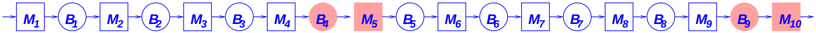
Infinite-Buffer Line





 Copyright ©2010 Stanley B. Gershwin.

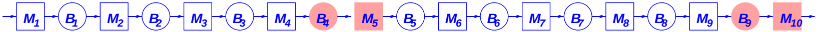
Infinite-Buffer Line

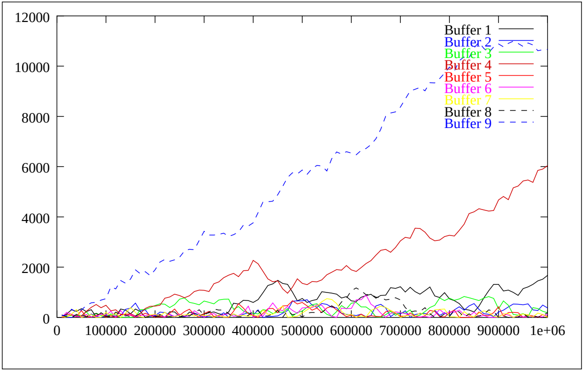


- The *second bottleneck* is the slowest machine upstream of the bottleneck. An infinite amount of inventory accumulates just upstream of it.
- A finite amount of inventory appears between the second bottleneck and the machine upstream of the first bottleneck.
- Et cetera.

 Copyright ©2010 Stanley B. Gershwin.

Infinite-Buffer Line

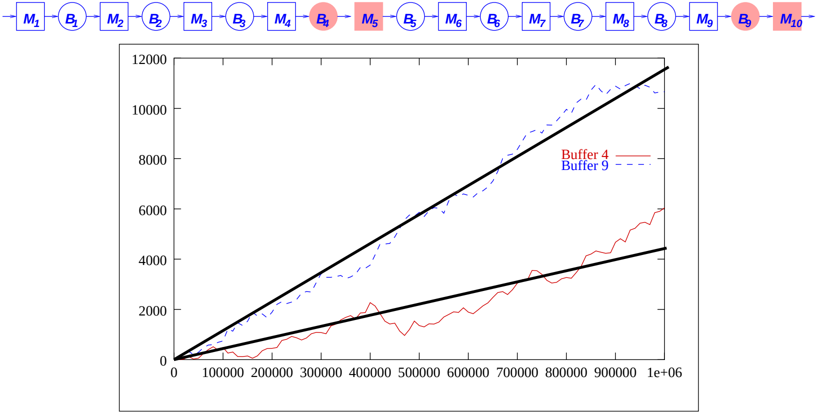




A 10-machine line with bottlenecks at Machines 5 and 10.

 Copyright ©2010 Stanley B. Gershwin.

Infinite-Buffer Line



Question:

- What are the slopes *(roughly!)* of the two indicated graphs?

 Copyright ©2010 Stanley B. Gershwin.

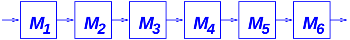
Infinite-Buffer Line

Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

 Copyright ©2010 Stanley B. Gershwin.

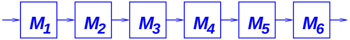
Zero-Buffer Line



- If any one machine fails, or takes a very long time to do an operation, *all* the other machines must wait.
- Therefore the production rate is usually less — possibly much less – than the slowest machine.

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line

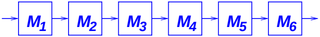


- *Special case:* Constant, unequal operation times, perfectly reliable machines.
  - The operation time of the line is equal to the operation time of the slowest machine, so the production rate of the line is *equal to* that of the slowest machine.

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line

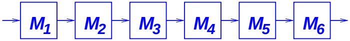
Constant, equal operation times, unreliable machines



- *Assumption:* Failure and repair times are *geometrically* distributed.
  - Define $p_i = \tau/\text{MTTF}_i$ = probability of failure during an operation.
  - Define $r_i = \tau/\text{MTTR}_i$ probability of repair during an interval of length τ when the machine is down.
- *Operation-Dependent Failures* (ODFs): Machines can only fail while they are working.

 Copyright © 2010 Stanley B. Gershwin.

Zero-Buffer Line



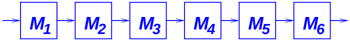
Buzacott's Zero-Buffer Line Formula:

Let k be the number of machines in the line. Then

$$P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line



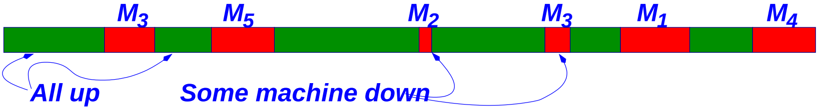
- Same as the earlier formula (page 11, page 14) when $k = 1$. The *isolated production rate* of a single machine M_i is

$$\frac{1}{\tau}\left(\frac{1}{1+\frac{p_i}{r_i}}\right) = \frac{1}{\tau}\left(\frac{r_i}{r_i+p_i}\right).$$

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line
Proof of formula

- Let τ (the operation time) be the time unit.
- *Assumption:* At most, <u>one</u> machine can be down.
- Consider a long time interval of length $T\tau$ during which Machine M_i fails m_i times ($i = 1, \ldots k$).



- Without failures, the line would produce T parts.

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line

- The average repair time of M_i is τ / r_i each time it fails, so the total system down time is close to

$$D\tau = \sum_{i=1}^{k} \frac{m_i \tau}{r_i}$$

where D is the number of operation times in which a machine is down.

 Copyright © 2010 Stanley B. Gershwin.

Zero-Buffer Line

- The total up time is approximately

$$U\tau = T\tau - \sum_{i=1}^{k} \frac{m_i \tau}{r_i}.$$

- where U is the number of operation times in which all machines are up.

 Copyright © 2010 Stanley B. Gershwin.

Zero-Buffer Line

- Since the system produces one part per time unit while it is working, it produces U parts during the interval of length $T\tau$.
- Note that, approximately,

$$m_i = p_i U$$

because M_i can only fail while it is operational.

 Copyright © 2010 Stanley B. Gershwin.

Zero-Buffer Line

- Thus,

$$U\tau = T\tau - U\tau \sum_{i=1}^{k} \frac{p_i}{r_i},$$

or,

$$\frac{U}{T} = E_{ODF} = \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line

and

$$P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

- Note that P is a function of the *ratio* p_i/r_i and not p_i or r_i separately.
- The same statement is true for the infinite-buffer line.
- However, the same statement is *not* true for a line with finite, non-zero buffers.

 Copyright ©2010 Stanley B. Gershwin.

Zero-Buffer Line

Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

Copyright ©2010 Stanley B. Gershwin.

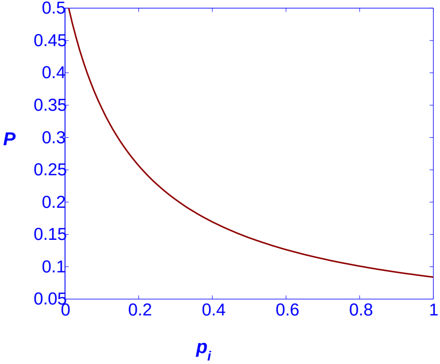
TDF= Time-Dependent Failure. Machines fail independently of one another when they are idle.

$$P_{TDF} = \frac{1}{\tau} \prod_{i=1}^{k} \left(\frac{r_i}{r_i + p_i} \right) > P_{ODF}$$

 Copyright ©2010 Stanley B. Gershwin.

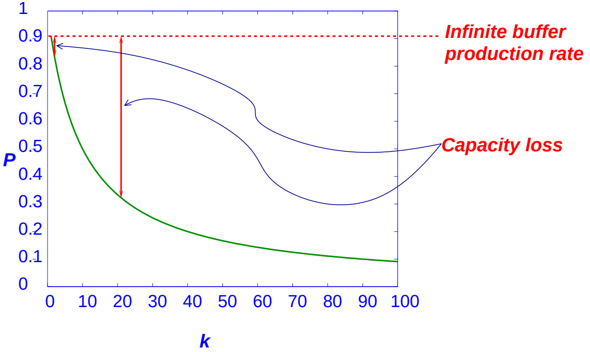
Zero-Buffer Line
P as a function of p_i

All machines are the same except M_i. As p_i increases, the production rate decreases.



Zero-Buffer Line
P as a function of *k*

All machines are the same. As the line gets longer, the production rate decreases.



 Copyright ©2010 Stanley B. Gershwin.

Finite-Buffer Lines

- Motivation for buffers: recapture some of the lost production rate.
- Cost
  - in-process inventory/lead time
  - floor space
  - material handling mechanism

Copyright ©2010 Stanley B. Gershwin.

Finite-Buffer Lines

- Infinite buffers: no propagation of disruptions.
- Zero buffers: instantaneous propagation.
- Finite buffers: delayed propagation.
  - New phenomena: *blockage* and *starvation* .

Copyright ©2010 Stanley B. Gershwin.

Finite-Buffer Lines

- Difficulty:
  - No simple formula for calculating production rate or inventory levels.
- Solution:
  - Simulation
  - Analytical approximation

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

- Exact solution *is* available to model of two-machine line.
- *Discrete time-discrete state Markov process:*

$$\text{prob}\{X(t+1) = x(t+1)| \\ X(t) = x(t), X(t-1) = x(t-1), X(t-2) = x(t-2), ...\} =$$

$$\text{prob}\{X(t+1) = x(t+1)|X(t) = x(t)\}$$

- In the following, we construct $\text{prob}\{X(t+1) = x(t+1)|X(t) = x(t)\}$ and solve the steady-state transition equations.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Here, $X(t) = (n(t), \alpha_1(t), \alpha_2(t))$, where

- n is the number of parts in the buffer; $n = 0, 1, ..., N$.
- α_i is the repair state of M_i; $i = 1, 2$.
  - $\alpha_i = 1$ means the machine is *up* or *operational*;
  - $\alpha_i = 0$ means the machine is *down* or *under repair*.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Motivation:

- We can develop intuition from these systems that is useful for understanding more complex systems.

- Two-machine lines are used as *building blocks* in decomposition approximations of realistic-sized systems.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Several models available:

- *Deterministic processing time,* or *Buzacott model:* deterministic processing time, geometric failure and repair times; discrete state, discrete time.
- *Exponential processing time:* exponential processing, failure, and repair time; discrete state, continuous time.
- *Continuous material,* or *fluid:* deterministic processing, exponential failure and repair time; mixed state, continuous time.
- *Extensions*
  - Models with multiple up and down states.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Outline: Details of two-machine, deterministic processing time line.

- Assumptions
- Performance measures
- Transient states
- Transition equations
- Identities
- Analytical solution
- Limits
- Behavior

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Assumptions, etc.

Assumptions, etc. for deterministic processing time systems (including long lines)

- All operation times are deterministic and equal to 1.
- The amount of material in Buffer i at time t is $n_i(t)$, $0 \leq n_i(t) \leq N_i$. A buffer gains or loses at most one piece during a time unit.
- The state of the system is $s = (n_1, \ldots, n_{k-1}, \alpha_1, \ldots, \alpha_k)$.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Assumptions, etc.

- *Operation dependent failures:*

$$\begin{aligned}
\text{prob}\left[\alpha_i(t+1) = 0 \mid n_{i-1}(t) = 0, \alpha_i(t) = 1, n_i(t) < N_i\right] &= 0,\\
\text{prob}\left[\alpha_i(t+1) = 1 \mid n_{i-1}(t) = 0, \alpha_i(t) = 1, n_i(t) < N_i\right] &= 1,
\end{aligned}$$

$$\begin{aligned}
\text{prob}\left[\alpha_i(t+1) = 0 \mid n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) = N_i\right] &= 0,\\
\text{prob}\left[\alpha_i(t+1) = 1 \mid n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) = N_i\right] &= 1,
\end{aligned}$$

$$\begin{aligned}
\text{prob}\left[\alpha_i(t+1) = 0 \mid n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i\right] &= p_i,\\
\text{prob}\left[\alpha_i(t+1) = 1 \mid n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i\right] &= 1 - p_i.
\end{aligned}$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Assumptions, etc.

- *Repairs:*

$$\text{prob } [\alpha_i(t+1) = 1 \mid \alpha_i(t) = 0] = r_i,$$

$$\text{prob } [\alpha_i(t+1) = 0 \mid \alpha_i(t) = 0] = 1 - r_i.$$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Assumptions, etc.

- *Timing convention:* In the absence of blocking or starvation:

$$n_i(t+1) = n_i(t) + \alpha_i(t+1) - \alpha_{i+1}(t+1).$$

More generally,

$$n_i(t+1) = n_i(t) + \mathcal{I}_{ui}(t+1) - \mathcal{I}_{di}(t+1),$$

where

$$\mathcal{I}_{ui}(t+1) = \begin{cases} 1 \text{ if } \alpha_i(t+1) = 1 \text{ and } n_{i-1}(t) > 0 \text{ and } n_i(t) < N_i, \\ 0 \text{ otherwise.} \end{cases}$$

$$\mathcal{I}_{di}(t+1) = \begin{cases} 1 \text{ if } \alpha_{i+1}(t+1) = 1 \text{ and } n_i(t) > 0 \text{ and } n_{i+1}(t) < N_{i+1} \\ 0 \text{ otherwise.} \end{cases}$$

 Copyright ©2010 Stanley B. Gershwin.

- In the Markov chain model, there is a set of transient states, and a single final class. Thus, a unique steady state distribution exists. The model is studied in steady state. That is, we calculate the stationary probability distribution.

- We calculate performance measures (production rate and average inventory) from the steady state distribution.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Performance measures

- The steady state *production rate* (*throughput*, *flow rate*, or *efficiency*) of Machine M_i is the probability that Machine M_i produces a part in a time step.
- Units: parts per operation time.
- It is the probability that Machine M_i is operational and neither starved nor blocked in time step t.
- It is equivalent, and more convenient, to express it as the probability that Machine M_i is operational and neither starved nor blocked in time step $t+1$:

$$E_i = \text{ prob } (\alpha_i(t+1) = 1, n_{i-1}(t) > 0, n_i(t) < N_i)$$

For a useful analytical expression, we must rewrite this so that all states are evaluated at the same time.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Performance measures

$$
\begin{aligned}
E_i &= \text{prob}\ (\alpha_i(t+1) = 1, n_{i-1}(t) > 0, n_i(t) < N_i) \\
&= \text{prob}\ (\alpha_i(t+1) = 1 \mid n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i) \\
&\qquad \text{prob}\ (n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i) \\
&+ \text{prob}\ (\alpha_i(t+1) = 1 \mid n_{i-1}(t) > 0, \alpha_i(t) = 0, n_i(t) < N_i) \\
&\qquad \text{prob}\ (n_{i-1}(t) > 0, \alpha_i(t) = 0, n_i(t) < N_i). \\
&= (1 - p_i)\ \text{prob}\ (n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i) \\
&\quad + r_i\ \text{prob}\ (n_{i-1}(t) > 0, \alpha_i(t) = 0, n_i(t) < N_i).
\end{aligned}
$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Performance measures

In steady state, there is a repair for every failure of Machine i, or

$$r_i \text{ prob } (n_{i-1}(t) > 0, \alpha_i(t) = 0, n_i(t) < N_i) =$$
$$p_i \text{ prob } (n_{i-1}(t) > 0, \alpha_i(t) = 1, n_i(t) < N_i)$$

Therefore,

$$E_i = \text{ prob } (\alpha_i = 1, n_{i-1} > 0, n_i < N_i).$$

Copyright ©2010 Stanley B. Gershwin.

The steady state average level of Buffer i is

$$\bar{n}_i = \sum_s n_i \text{ prob } (s).$$

 Copyright ©2010 Stanley B. Gershwin.

$$s = (n, \alpha_1, \alpha_2)$$

where

$$n = 0, 1, ..., N$$

$$\alpha_i = 0, 1$$

Two-Machine, Finite-Buffer Lines

Transient states

- (0,1,0) is transient because it cannot be reached from any state. If $\alpha_1(t+1) = 1$ and $\alpha_2(t+1) = 0$, then $n(t+1) = n(t) + 1$.

- (0,1,1) is transient because it cannot be reached from any state. If $n(t) = 0$ and $\alpha_1(t+1) = 1$ and $\alpha_2(t+1) = 1$, then $n(t+1) = 1$ since M_2 is starved and thus not able to operate. If $n(t) > 0$ and $\alpha_1(t+1) = 1$ and $\alpha_2(t+1) = 1$, then $n(t+1) = n(t)$.

Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Transient states

- (0,0,0) is transient because it can be reached only from itself or (0,1,0). It can be reached from itself if neither machine is repaired; it can be reached from (0,1,0) if the first machine fails while attempting to make a part. It cannot be reached from (0,0,1) or (0,1,1) since the second machine cannot fail. Otherwise, if $\alpha_1(t+1) = 0$ and $\alpha_2(t+1) = 0$, then $n(t+1) = n(t)$.

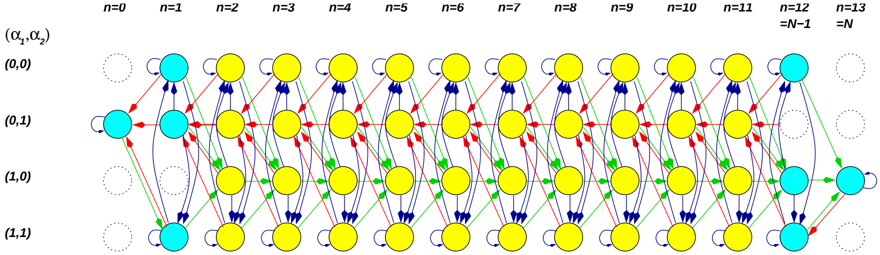
- (1,1,0) is transient because it can be reached only from (0,0,0) or (0,1,0). If $\alpha_1(t+1) = 1$ and $\alpha_2(t+1) = 0$, then $n(t+1) = n(t) + 1$. Therefore, $n(t) = 0$. However, (1,1,0) cannot be reached from (0,0,1) since Machine 2 cannot fail. (For the same reason, it cannot be reached from (0,1,1), but since the latter is transient, that is irrelevant.)

- Similarly, $(N, 0, 0)$, $(N, 0, 1), (N, 1, 1)$, and $(N - 1, 0, 1)$ are transient.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

State space



key

states

transient

non−transient

boundary

internal

transitions

out of transient states

out of non−transient states

to increasing buffer level

to decreasing buffer level

unchanging buffer level

Two-Machine, Finite-Buffer Lines

Transition equations

Internal equations $2 \le n \le N-2$

$$\begin{aligned}
\mathbf{p}(n,0,0) = \; & (1-r_1)(1-r_2)\mathbf{p}(n,0,0) + (1-r_1)p_2\mathbf{p}(n,0,1) \\
& + p_1(1-r_2)\mathbf{p}(n,1,0) + p_1p_2\mathbf{p}(n,1,1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(n,0,1) = \; & (1-r_1)r_2\mathbf{p}(n+1,0,0) + (1-r_1)(1-p_2)\mathbf{p}(n+1,0,1) \\
& + p_1r_2\mathbf{p}(n+1,1,0) + p_1(1-p_2)\mathbf{p}(n+1,1,1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(n,1,0) = \; & r_1(1-r_2)\mathbf{p}(n-1,0,0) + r_1p_2\mathbf{p}(n-1,0,1) \\
& + (1-p_1)(1-r_2)\mathbf{p}(n-1,1,0) + (1-p_1)p_2\mathbf{p}(n-1,1,1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(n,1,1) = \; & r_1r_2\mathbf{p}(n,0,0) + r_1(1-p_2)\mathbf{p}(n,0,1) + (1-p_1)r_2\mathbf{p}(n,1,0) \\
& + (1-p_1)(1-p_2)\mathbf{p}(n,1,1)
\end{aligned}$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Transition equations

Lower boundary equations $n \leq 1$

$$\begin{aligned}\mathbf{p}(0,0,1) = \; & (1-r_1)\mathbf{p}(0,0,1) + (1-r_1)r_2\mathbf{p}(1,0,0) \\ & +(1-r_1)(1-p_2)\mathbf{p}(1,0,1) + p_1(1-p_2)\mathbf{p}(1,1,1).\end{aligned}$$

$$\mathbf{p}(1,0,0) = (1-r_1)(1-r_2)\mathbf{p}(1,0,0) + (1-r_1)p_2\mathbf{p}(1,0,1) + p_1 p_2\mathbf{p}(1,1,1)$$

$$\begin{aligned}\mathbf{p}(1,0,1) = \; & (1-r_1)r_2\mathbf{p}(2,0,0) + (1-r_1)(1-p_2)\mathbf{p}(2,0,1) + \\ & p_1 r_2\mathbf{p}(2,1,0) + p_1(1-p_2)\mathbf{p}(2,1,1)\end{aligned}$$

$$\begin{aligned}\mathbf{p}(1,1,1) = \; & r_1\mathbf{p}(0,0,1) + r_1 r_2\mathbf{p}(1,0,0) + r_1(1-p_2)\mathbf{p}(1,0,1) \\ & +(1-p_1)(1-p_2)\mathbf{p}(1,1,1)\end{aligned}$$

$$\mathbf{p}(2,1,0) = r_1(1-r_2)\mathbf{p}(1,0,0) + r_1 p_2\mathbf{p}(1,0,1) + (1-p_1)p_2\mathbf{p}(1,1,1)$$

Two-Machine, Finite-Buffer Lines

Transition equations

Upper boundary equations $n \geq N-1$

$$\begin{aligned}
\mathbf{p}(N-2,0,1) = \; & (1-r_1)r_2\mathbf{p}(N-1,0,0) + p_1 r_2 \mathbf{p}(N-1,1,0) \\
& + p_1(1-p_2)\mathbf{p}(N-1,1,1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(N-1,0,0) = \; & (1-r_1)(1-r_2)\mathbf{p}(N-1,0,0) + p_1(1-r_2)\mathbf{p}(N-1,1,0) \\
& + p_1 p_2 \mathbf{p}(N-1,1,1)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(N-1,1,0) = \; & r_1(1-r_2)\mathbf{p}(N-2,0,0) + r_1 p_2 \mathbf{p}(N-2,0,1) \\
+ \; & (1-p_1)(1-r_2)\mathbf{p}(N-2,1,0) + (1-p_1)p_2\mathbf{p}(N-2,1,1)
\end{aligned}$$

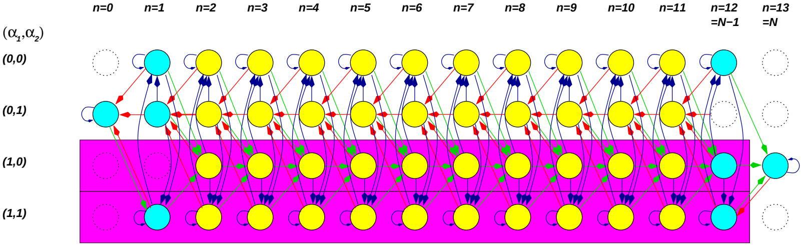
$$\begin{aligned}
\mathbf{p}(N-1,1,1) = \; & r_1 r_2 \mathbf{p}(N-1,0,0) + (1-p_1)r_2\mathbf{p}(N-1,1,0) \\
& + (1-p_1)(1-p_2)\mathbf{p}(N-1,1,1) + r_2\mathbf{p}(N,1,0)
\end{aligned}$$

$$\begin{aligned}
\mathbf{p}(N,1,0) = \; & r_1(1-r_2)\mathbf{p}(N-1,0,0) + (1-p_1)(1-r_2)\mathbf{p}(N-1,1,0) \\
& + (1-p_1)p_2\mathbf{p}(N-1,1,1) + (1-r_2)\mathbf{p}(N,1,0)
\end{aligned}$$

Copyright ©2010 Stanley B. Gershwin.

E_1 is the probability that M_1 is operational and not blocked:

$$E_1 = \sum_{\substack{n < N \\ \alpha_1 = 1}} \mathbf{p}(n, \alpha_1, \alpha_2).$$

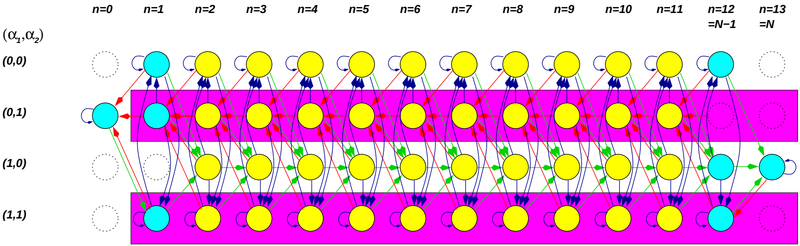


 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Performance measures

E_2 is the probability that M_2 is operational and not starved:

$$E_2 = \sum_{\substack{n>0 \\ \alpha_2 = 1}} \mathbf{p}(n, \alpha_1, \alpha_2).$$



 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Performance measures

The probabilities of starvation and blockage are:

$$p_s = \mathbf{p}(0,0,1), \text{ the probability of starvation,}$$

$$p_b = \mathbf{p}(N,1,0), \text{ the probability of blockage.}$$

The average buffer level is:

$$\bar{n} = \sum_{\text{all } s} n\mathbf{p}(n, \alpha_1, \alpha_2).$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Identities

Repair frequency equals failure frequency For every repair, there is a failure (in steady state). When the system is in steady state,

$$r_1 \text{ prob } [\{\alpha_1 = 0\} \text{ and } \{n < N\}] = \\ p_1 \text{ prob } [\{\alpha_1 = 1\} \text{ and } \{n < N\}] .$$

Let

$$D_1 = \text{ prob } [\{\alpha_1 = 0\} \text{ and } \{n < N\}] ,$$

then

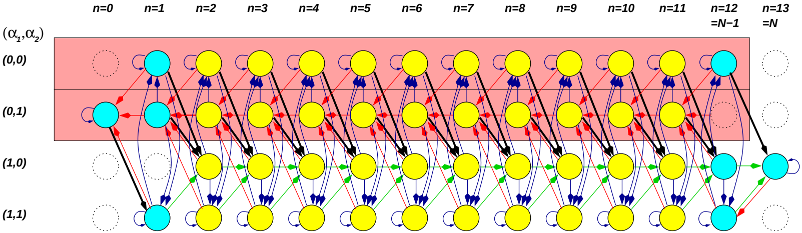
$$r_1 D_1 = p_1 E_1 .$$

Copyright ©2010 Stanley B. Gershwin.

Proof: The left side is the probability that the state leaves the set of states

$$\mathcal{S}_0 = \{\{\alpha_1 = 0\} \text{ and } \{n < N\}\}.$$

since the only way the system can leave $\mathcal{S}_0$ is for M_1 to get repaired. (M_1 is down, so the buffer cannot become full.)



 Copyright ©2010 Stanley B. Gershwin.

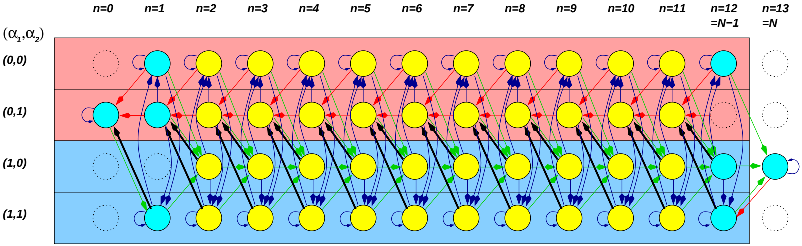
Two-Machine, Finite-Buffer Lines

Identities

The right side is the probability that the state enters $\mathcal{S}_0$. When the system is in steady state, the only way for the state to enter $\mathcal{S}_0$ is for it to be in set

$$\mathcal{S}_1 = \{\{\alpha_1 = 1\} \text{ and } \{n < N\}\}$$

in the previous time unit.



 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Identities

Conservation of Flow $E_1 = E_2 = E$

Proof:

$$E_1 = \sum_{n=0}^{N-1} \mathbf{p}(n,1,0) + \sum_{n=0}^{N-1} \mathbf{p}(n,1,1),$$

$$E_2 = \sum_{n=1}^{N} \mathbf{p}(n,0,1) + \sum_{n=1}^{N} \mathbf{p}(n,1,1).$$

Then

$$E_1 - E_2 = \sum_{n=0}^{N-1} \mathbf{p}(n,1,0) - \sum_{n=1}^{N} \mathbf{p}(n,0,1)$$

$$= \sum_{n=1}^{N-2} \mathbf{p}(n+1,1,0) - \sum_{n=1}^{N-2} \mathbf{p}(n,0,1)$$

Copyright ©2010 Stanley B. Gershwin.

Or,

$$E_1 - E_2 = \sum_{n=1}^{N-2} \Big(\mathbf{p}(n+1,1,0) - \mathbf{p}(n,0,1) \Big)$$

Define $\delta(n) = \mathbf{p}(n+1,1,0) - \mathbf{p}(n,0,1)$. Then

$$E_1 - E_2 = \sum_{n=1}^{N-2} \delta(n)$$

Two-Machine, Finite-Buffer Lines

Identities

Add lots of lower boundary equations:

$$\mathbf{p}(0,0,1) + \mathbf{p}(1,0,0) + \mathbf{p}(1,1,1) + \mathbf{p}(2,1,0) =$$

$$(1 - r_1)\mathbf{p}(0,0,1) + (1 - r_1)r_2\mathbf{p}(1,0,0)$$
$$+(1 - r_1)(1 - p_2)\mathbf{p}(1,0,1) + p_1(1 - p_2)\mathbf{p}(1,1,1)$$

$$+(1 - r_1)(1 - r_2)\mathbf{p}(1,0,0) + (1 - r_1)p_2\mathbf{p}(1,0,1) + p_1 p_2\mathbf{p}(1,1,1)$$

$$+r_1\mathbf{p}(0,0,1) + r_1 r_2\mathbf{p}(1,0,0) + r_1(1 - p_2)\mathbf{p}(1,0,1)$$
$$+(1 - p_1)(1 - p_2)\mathbf{p}(1,1,1)$$

$$+r_1(1 - r_2)\mathbf{p}(1,0,0) + r_1 p_2\mathbf{p}(1,0,1) + (1 - p_1)p_2\mathbf{p}(1,1,1)$$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Identities

Or,

$$\mathbf{p}(0,0,1)+\mathbf{p}(1,0,0)+\mathbf{p}(1,1,1)+\mathbf{p}(2,1,0) =$$

$$\mathbf{p}(0,0,1)+\mathbf{p}(1,0,0)+\mathbf{p}(1,0,1)+\mathbf{p}(1,1,1)$$

Or,

$$\mathbf{p}(2,1,0) = \mathbf{p}(1,0,1)$$

Then $\delta(1) = 0$.

Two-Machine, Finite-Buffer Lines

Identities

Now add all the internal equations, after changing the index of two of them:

$$\mathbf{p}(n,0,0) + \mathbf{p}(n-1,0,1) + \mathbf{p}(n+1,1,0) + \mathbf{p}(n,1,1) =$$

$$\begin{array}{c}(1-r_1)(1-r_2)\mathbf{p}(n,0,0) + (1-r_1)p_2\mathbf{p}(n,0,1) \\ +p_1(1-r_2)\mathbf{p}(n,1,0) + p_1 p_2\mathbf{p}(n,1,1)\end{array}$$

$$\begin{array}{c}(1-r_1)r_2\mathbf{p}(n,0,0) + (1-r_1)(1-p_2)\mathbf{p}(n,0,1) \\ +p_1 r_2\mathbf{p}(n,1,0) + p_1(1-p_2)\mathbf{p}(n,1,1)\end{array}$$

$$\begin{array}{c}r_1(1-r_2)\mathbf{p}(n,0,0) + r_1 p_2\mathbf{p}(n,0,1) \\ +(1-p_1)(1-r_2)\mathbf{p}(n,1,0) + (1-p_1)p_2\mathbf{p}(n,1,1)\end{array}$$

$$\begin{array}{c}r_1 r_2\mathbf{p}(n,0,0) + r_1(1-p_2)\mathbf{p}(n,0,1) + (1-p_1)r_2\mathbf{p}(n,1,0) \\ +(1-p_1)(1-p_2)\mathbf{p}(n,1,1)\end{array}$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Identities

Or, for $n = 2, \ldots, N-2$,

$$\mathbf{p}(n,0,0) \;+\; \mathbf{p}(n-1,0,1) \;+\; \mathbf{p}(n+1,1,0) \;+\; \mathbf{p}(n,1,1) =$$

$$\mathbf{p}(n,0,0) \;+\; \mathbf{p}(n,0,1) \;+\; \mathbf{p}(n,1,0) \;+\; \mathbf{p}(n,1,1),$$

or,

$$\mathbf{p}(n+1,1,0) - \mathbf{p}(n,0,1) = \mathbf{p}(n,1,0) - \mathbf{p}(n-1,0,1)$$

or,

$$\delta(n) = \delta(n-1)$$

Since

$$\delta(1) = 0 \quad \text{and} \quad \delta(n) = \delta(n-1), n = 2, \ldots, N-2$$

we have

$$\delta(n) = 0, \qquad n = 1, \ldots, N-2$$

Therefore

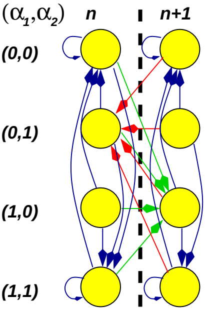
$$E_1 - E_2 = \sum_{n=1}^{N-2} \delta(n) = 0$$

QED

Two-Machine, Finite-Buffer Lines

Identities

Alternative interpretation of $\mathbf{p}(n+1,1,0) - \mathbf{p}(n,0,1) = 0$*:*



- The only way the buffer can go from $n+1$ to n is for the state to go to $(n,0,1)$.
- The only way the buffer can go from n to $n+1$ is for the state to go to $(n+1,1,0)$.

Flow rate/idle time

$$E = e_1(1 - p_b).$$

Proof: From the definitions of E_1 and D_1, we have

$$\text{prob } [n < N] = E + D_1,$$

or,

$$1 - p_b = E + \frac{p_1}{r_1} E = \frac{E}{e_1}.$$

Similarly,

$$E = e_2(1 - p_s)$$

1. Guess a solution for the internal states of the form $\mathbf{p}(n, \alpha_1, \alpha_2) = \xi_j(n, \alpha_1, \alpha_2) = X^n Y_1^{\alpha_1} Y_2^{\alpha_2}$.
2. Determine sets of X_j, Y_{1j}, Y_{2j} that satisfy the internal equations.
3. Extend $\xi_j(n, \alpha_1, \alpha_2)$ to *all* of the boundary states using *some* of the boundary equations.
4. Find coefficients C_j so that $\mathbf{p}(n, \alpha_1, \alpha_2) = \sum_j C_j \xi_j(n, \alpha_1, \alpha_2)$ satisfies the remaining boundary equationss and normalization.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Internal equations:

$$X^n = (1-r_1)(1-r_2)X^n + (1-r_1)p_2X^nY_2 + p_1(1-r_2)X^nY_1 + p_1p_2X^nY_1Y_2$$

$$\begin{aligned} X^{n-1}Y_2 = \quad & (1-r_1)r_2X^n + (1-r_1)(1-p_2)X^nY_2 + p_1r_2X^nY_1 \\ & + p_1(1-p_2)X^nY_1Y_2 \end{aligned}$$

$$X^{n+1}Y_1 = r_1(1-r_2)X^n + r_1p_2X^nY_2 + (1-p_1)(1-r_2)X^nY_1 + (1-p_1)p_2X^nY_1Y_2$$

$$X^nY_1Y_2 = r_1r_2X^n + r_1(1-p_2)X^nY_2 + (1-p_1)r_2X^nY_1 + (1-p_1)(1-p_2)X^nY_1Y_2$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Or,

$$1 = (1 - r_1)(1 - r_2) + (1 - r_1)p_2 Y_2 + p_1(1 - r_2)Y_1 + p_1 p_2 Y_1 Y_2$$

$$X^{-1} Y_2 = \begin{array}{l}(1 - r_1)r_2 + (1 - r_1)(1 - p_2)Y_2 + p_1 r_2 Y_1 \\ +p_1(1 - p_2)Y_1 Y_2\end{array}$$

$$XY_1 = r_1(1 - r_2) + r_1 p_2 Y_2 + (1 - p_1)(1 - r_2)Y_1 + (1 - p_1)p_2 Y_1 Y_2$$

$$Y_1 Y_2 = r_1 r_2 + r_1(1 - p_2)Y_2 + (1 - p_1)r_2 Y_1 + (1 - p_1)(1 - p_2)Y_1 Y_2$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Or,

$$1 = (1 - r_1 + Y_1 p_1)\,(1 - r_2 + Y_2 p_2)$$

$$X^{-1} Y_2 = (1 - r_1 + Y_1 p_1)\,(r_2 + Y_2(1 - p_2))$$

$$X Y_1 = (r_1 + Y_1(1 - p_1))\,(1 - r_2 + Y_2 p_2)$$

$$Y_1 Y_2 = (r_1 + Y_1(1 - p_1))\,(r_2 + Y_2(1 - p_2))$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Since the last equation is a product of the other three, there are only three independent equations in three unknowns here. They may be simplified further:

$$1 = (1 - r_1 + Y_1 p_1)(1 - r_2 + Y_2 p_2)$$

$$XY_1 = \frac{r_1 + Y_1(1 - p_1)}{1 - r_1 + Y_1 p_1}$$

$$X^{-1} Y_2 = \frac{r_2 + Y_2(1 - p_2)}{1 - r_2 + Y_2 p_2}$$

 Copyright ©2010 Stanley B. Gershwin.

Eliminating X and Y_2, this becomes

$$
\begin{aligned}
0 \quad = \quad & Y_1^2 \left(p_1 + p_2 - p_1 p_2 - p_1 r_2 \right) \\
- \quad & Y_1 \left(r_1 \left(p_1 + p_2 - p_1 p_2 - p_1 r_2 \right) + p_1 \left(r_1 + r_2 - r_1 r_2 - r_1 p_2 \right) \right) \\
& + r_1 \left(r_1 + r_2 - r_1 r_2 - r_1 p_2 \right),
\end{aligned}
$$

which has two solutions:

$$
Y_{11} = \frac{r_1}{p_1}, \qquad Y_{12} = \frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{p_1 + p_2 - p_1 p_2 - p_1 r_2}.
$$

The complete solutions are:

$$
\left.\begin{aligned}
Y_{11} &= \frac{r_1}{p_1} \\
Y_{21} &= \frac{r_2}{p_2} \\
X_1 &= 1
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
Y_{12} &= \frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{p_1 + p_2 - p_1 p_2 - p_1 r_2} \\
Y_{22} &= \frac{r_1 + r_2 - r_1 r_2 - p_1 r_2}{p_1 + p_2 - p_1 p_2 - p_2 r_1} \\
X_2 &= \frac{Y_{22}}{Y_{12}}
\end{aligned}\right\}
$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Recall that $\xi(n, \alpha_1, \alpha_2) = X^n Y_1^{\alpha_1} Y_2^{\alpha_2}$.

We now have the complete *internal* solution:

$$\mathbf{p}(n, \alpha_1, \alpha_2) = C_1 \xi_1(n, \alpha_1, \alpha_2) + C_2 \xi_2(n, \alpha_1, \alpha_2)$$

$$= C_1 X_1^n Y_{11}^{\alpha_1} Y_{21}^{\alpha_2} + C_2 X_2^n Y_{12}^{\alpha_1} Y_{22}^{\alpha_2}.$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Boundary conditions:

If we plug the internal expression for $\xi(n, \alpha_1, \alpha_2) = X^n Y_1^{\alpha_1} Y_2^{\alpha_2}$ into the right side of

$$\begin{aligned} \xi(1,0,1) = \; & (1-r_1)r_2\xi(2,0,0) + (1-r_1)(1-p_2)\xi(2,0,1) + \\ & p_1 r_2 \xi(2,1,0) + p_1(1-p_2)\xi(2,1,1), \end{aligned}$$

we find

$$\xi(1,0,1) = XY_2$$

which implies that

$$\mathbf{p}(1,0,1) = C_1 Y_{21} + C_2 X_2 Y_{22}.$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Recall that

$$\mathbf{p}(2,1,0) = \mathbf{p}(1,0,1).$$

Then

$$C_1 X_1^2 Y_{11} + C_2 X_2^2 Y_{12} = C_1 X_1 Y_{21} + C_2 X_2 Y_{22},$$

or,

$$\left(C_1 X_1^2 Y_{11} - C_1 X_1 Y_{21} \right) + \left(C_2 X_2^2 Y_{12} - C_2 X_2 Y_{22} \right) = 0,$$

or,

$$C_1 X_1 \left(X_1 Y_{11} - Y_{21} \right) + C_2 X_2 \left(X_2 Y_{12} - Y_{22} \right) = 0,$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Recall

$$X_2 = \frac{Y_{22}}{Y_{12}}$$

Consequently,

$$C_1 X_1 \left(X_1 Y_{11} - Y_{21} \right) = 0,$$

or,

$$C_1 \left(\frac{r_1}{p_1} - \frac{r_2}{p_2} \right) = 0,$$

Therefore,

$$\text{if } \frac{r_1}{p_1} \neq \frac{r_2}{p_2}, \text{ then } C_1 = 0.$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

In the following, we assume $\frac{r_1}{p_1} \neq \frac{r_2}{p_2}$ and we drop the j subscript.

But what happens when $\frac{r_1}{p_1} = \frac{r_2}{p_2}$*?*

And what does $\frac{r_1}{p_1} = \frac{r_2}{p_2}$ *mean?*

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Combining the following two boundary conditions ...

$$
\begin{aligned}
r_1 \mathbf{p}(0,0,1) &= (1-r_1) r_2 \mathbf{p}(1,0,0) + (1-r_1)(1-p_2)\mathbf{p}(1,0,1) \\
&\quad + p_1(1-p_2)\mathbf{p}(1,1,1).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p}(1,1,1) = \; & r_1 \mathbf{p}(0,0,1) + r_1 r_2 \mathbf{p}(1,0,0) + r_1(1-p_2)\mathbf{p}(1,0,1) \\
& +(1-p_1)(1-p_2)\mathbf{p}(1,1,1)
\end{aligned}
$$

gives

$$\mathbf{p}(1,1,1) = r_2 \mathbf{p}(1,0,0) + (1-p_2) C X Y_2 + (1-p_2)\mathbf{p}(1,1,1)$$

or,

$$p_2 \mathbf{p}(1,1,1) = r_2 \mathbf{p}(1,0,0) + (1-p_2) C X Y_2 .$$

There are three unknown quantities: $\mathbf{p}(1,0,0)$, $\mathbf{p}(1,1,1)$, and C.

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Another boundary condition,

$$\mathbf{p}(1,0,0) = (1-r_1)(1-r_2)\mathbf{p}(1,0,0) + (1-r_1)p_2\mathbf{p}(1,0,1) + p_1p_2\mathbf{p}(1,1,1)$$

can be written

$$(r_1 + r_2 - r_1r_2)\mathbf{p}(1,0,0) = (1-r_1)p_2CXY_2 + p_1p_2\mathbf{p}(1,1,1).$$

which also has three unknown quantities: $\mathbf{p}(1,0,0)$, $\mathbf{p}(1,1,1)$, and C. If we eliminate $\mathbf{p}(1,1,1)$ and simplify, we get

$$(r_1 + r_2 - r_1r_2 - p_1r_2)\mathbf{p}(1,0,0) = (p_1 + p_2 - p_1p_2 - p_2r_1)CXY_2.$$

From the definition of Y_{22} (slide 85),

$$\mathbf{p}(1,0,0) = CX.$$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

If we plug this into the last equation on slide 91, we get

$$p_2 \mathbf{p}(1,1,1) = CX(r_2 + (1-p_2)Y_2)$$

or

$$\mathbf{p}(1,1,1) = \frac{CX}{p_2} \frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{p_1 + p_2 - p_1 p_2 - r_1 p_2}.$$

Finally, the first equation on slide 91 gives

$$\mathbf{p}(0,0,1) = CX \frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{r_1 p_2}.$$

The upper boundary conditions are determined in the same way.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Summary of Steady-State Probabilities: *Boundary values*

$$
\begin{aligned}
\mathbf{p}(0,0,0) &= 0 \\
\mathbf{p}(0,0,1) &= CX\frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{r_1 p_2} \\
\mathbf{p}(0,1,0) &= 0 \\
\mathbf{p}(0,1,1) &= 0 \\
\mathbf{p}(1,0,0) &= CX \\
\mathbf{p}(1,0,1) &= CXY_2 \\
\mathbf{p}(1,1,0) &= 0 \\
\mathbf{p}(1,1,1) &= \frac{CX}{p_2}\,\frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{p_1 + p_2 - p_1 p_2 - r_1 p_2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{p}(N-1,0,0) &= CX^{N-1} \\
\mathbf{p}(N-1,0,1) &= 0 \\
\mathbf{p}(N-1,1,0) &= CX^{N-1}Y_1 \\
\mathbf{p}(N-1,1,1) &= \frac{CX^{N-1}}{p_1}\,\frac{r_1 + r_2 - r_1 r_2 - p_1 r_2}{p_1 + p_2 - p_1 p_2 - p_1 r_2} \\
\mathbf{p}(N,0,0) &= 0 \\
\mathbf{p}(N,0,1) &= 0 \\
\mathbf{p}(N,1,0) &= CX^{N-1}\frac{r_1 + r_2 - r_1 r_2 - p_1 r_2}{p_1 r_2} \\
\mathbf{p}(N,1,1) &= 0
\end{aligned}
$$

Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Analytical Solution

Summary of Steady-State Probabilities: *Internal states, etc.*

$$\mathbf{p}(n, \alpha_1, \alpha_2) = C X^n Y_1^{\alpha_1} Y_2^{\alpha_2},$$

$$2 \leq n \leq N-2; \quad \alpha_1 = 0,1; \quad \alpha_2 = 0,1$$

where

$$\begin{aligned} Y_1 &= \frac{r_1 + r_2 - r_1 r_2 - r_1 p_2}{p_1 + p_2 - p_1 p_2 - p_1 r_2} \\ Y_2 &= \frac{r_1 + r_2 - r_1 r_2 - p_1 r_2}{p_1 + p_2 - p_1 p_2 - r_1 p_2} \\ X &= \frac{Y_2}{Y_1} \end{aligned}$$

and C is a normalizing constant.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Analytical Solution

Observations:

Typically, we can expect that $r_i < .2$ since a repair is likely to take at least 5 times as long as an operation. Also, since, typically, efficiency $= r_i/(r_i + p_i) > .7$, $p_i < .4r_i$, $\mathbf{p}(0,0,1)$, $\mathbf{p}(1,1,1)$, $\mathbf{p}(N-1,1,1)$, $\mathbf{p}(N,1,0)$ are much larger than internal probabilities.

This is because the system tends to spend much more time at those states than at internal states.

Refer to transition graph on page 60 to trace out typical scenarios.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Limits

If $r_1 \longrightarrow 0$, then $E \longrightarrow 0, p_s \longrightarrow 1, p_b \longrightarrow 0, \bar{n} \longrightarrow 0.$

If $r_2 \longrightarrow 0$, then $E \longrightarrow 0, p_b \longrightarrow 1, p_s \longrightarrow 0, \bar{n} \longrightarrow N.$

If $p_1 \longrightarrow 0$, then $p_s \longrightarrow 0, E \longrightarrow 1 - p_b \longrightarrow e_2, \bar{n} \longrightarrow N - e_2.$

If $p_2 \longrightarrow 0$, then $p_b \longrightarrow 0, E \longrightarrow 1 - p_s \longrightarrow e_1, \bar{n} \longrightarrow e_1.$

If $N \longrightarrow \infty$ and $e_1 < e_2$, then $E \longrightarrow e_1, p_b \longrightarrow 0, p_s \longrightarrow 1 - \dfrac{e_1}{e_2}.$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Limits

Proof:

Many of the limits follow from combining conservation of flow and the flow rate-idle time relationship:

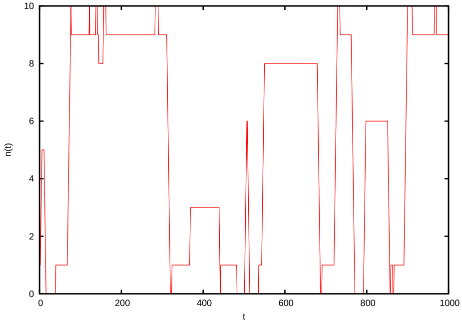
$$E = \frac{r_1}{r_1 + p_1}(1 - p_b) = \frac{r_2}{r_2 + p_2}(1 - p_s).$$

The last set comes from the analytic solution and the observation that if $e_1 > e_2, \quad X > 1$, and if $e_1 < e_2, \quad X < 1$.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

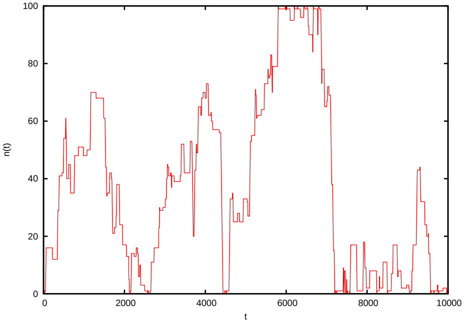
Behavior



$$r_1 = .1, p_1 = .01, r_2 = .1, p_2 = .01, N = 10$$

Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Behavior

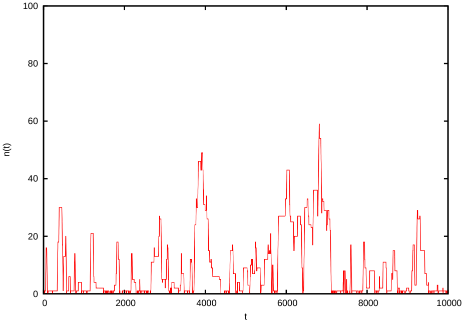


$$r_1 = .1, p_1 = .01, r_2 = .1, p_2 = .01, N = 100$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

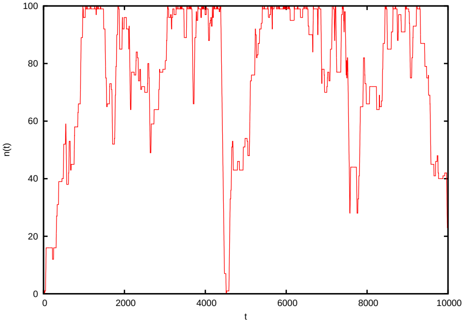
Behavior



$$r_i = .1, i = 1, 2, p_1 = .02, p_2 = .01, N = 100$$

Copyright ©2010 Stanley B. Gershwin.

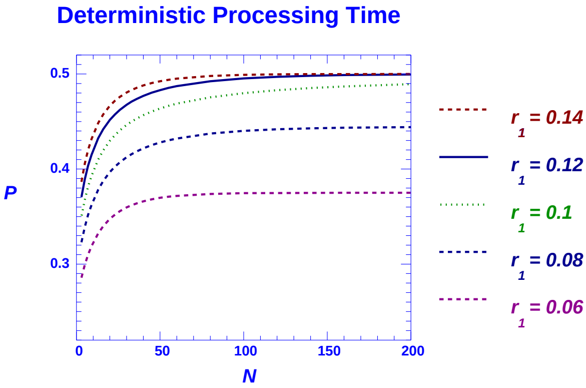
Two-Machine, Finite-Buffer Lines
Behavior



$$r_i = .1, i = 1, 2, p_1 = .01, p_2 = .02, N = 100$$

Copyright © 2010 Stanley B. Gershwin.

$\tau = 1.$
$p_1 = .1$
$r_2 = .1$
$p_2 = .1$



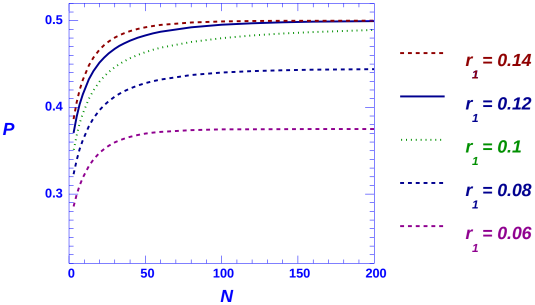
 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Behavior

Discussion:

- Why are the curves increasing?
- Why do they reach an asymptote?
- What is P when $N = 0$?
- What is the limit of P as $N \to \infty$?
- Why are the curves with smaller r_1 lower?

Deterministic Processing Time

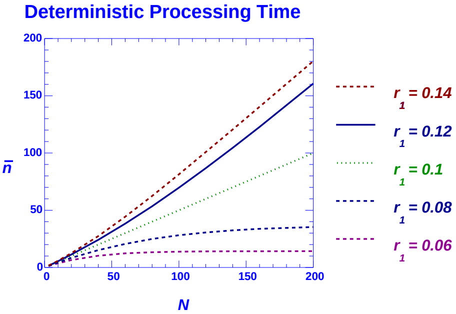


 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Behavior

Discussion:

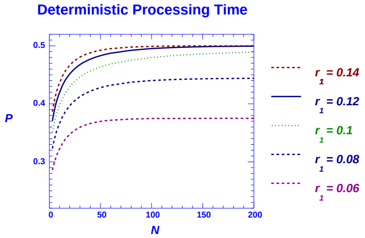
- Why are the curves increasing?
- Why *different* asymptotes?
- What is $\bar{n}$ when $N = 0$?
- What is the limit of $\bar{n}$ as $N \to \infty$?
- Why are the curves with smaller r_1 lower?

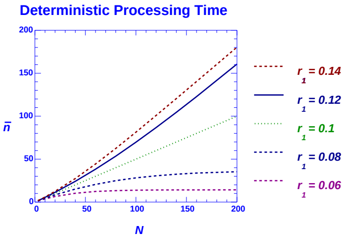


 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Behavior





- *What can you say about the optimal buffer size?*
- *How should it be related to r_i, p_i?*

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Behavior

Questions:

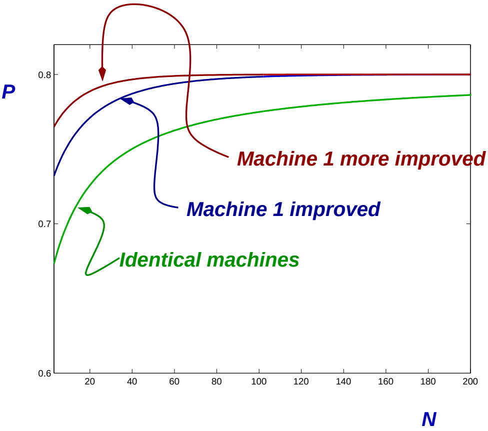
- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Production rate vs. storage space

Improvements to *non-bottleneck* machine.



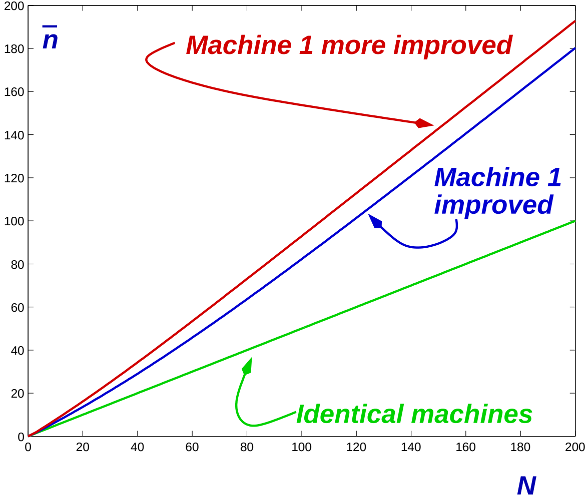
Note: Graphs would be the same if we improved Machine 2.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Average inventory vs. storage space

Inventory *increases* as the (non-bottleneck) *upstream* machine is improved and as the buffer space is increased.

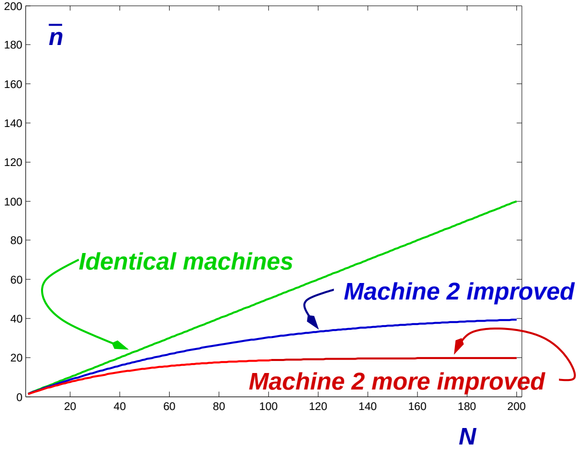


 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Average inventory vs. storage space

- Inventory *decreases* as the (non-bottleneck) *downstream* machine is improved.
- Inventory *increases* as the buffer space is increased.



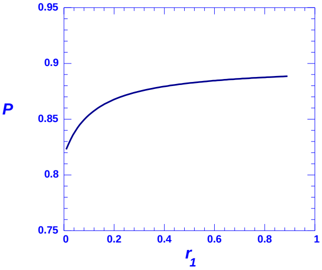
 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Frequency and Production Rate

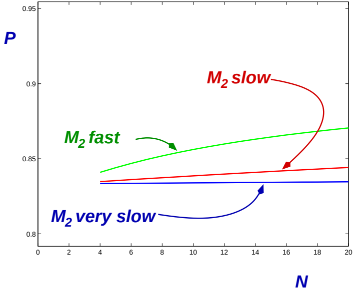
Should we prefer short, frequent, disruptions or long, infrequent, disruptions?

- $r_2 = 0.8$, $p_2 = 0.09$, $N = 10$
- r_1 and p_1 vary together and $\frac{r_1}{r_1 + p_1} = .9$
- *Answer:* evidently, short, frequent failures.
- *Why?*



 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Frequency and Production Rate

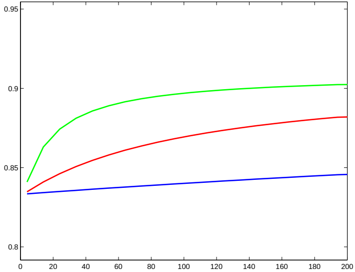


- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

 Copyright ©2010 Stanley B. Gershwin.

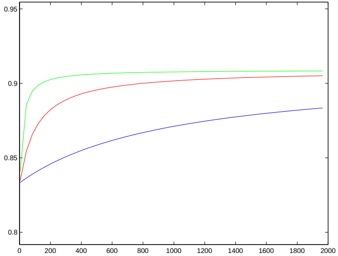
Two-Machine, Finite-Buffer Lines

Frequency and Production Rate



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

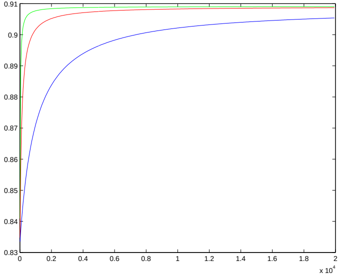
 Copyright ©2010 Stanley B. Gershwin.



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

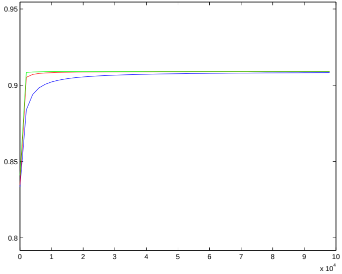
Two-Machine, Finite-Buffer Lines

Frequency and Production Rate



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

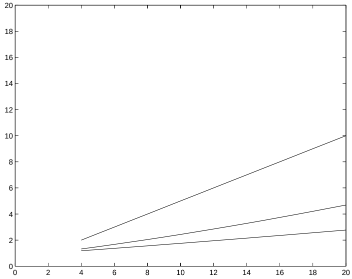
 Copyright © 2010 Stanley B. Gershwin.



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

Copyright ©2010 Stanley B. Gershwin.

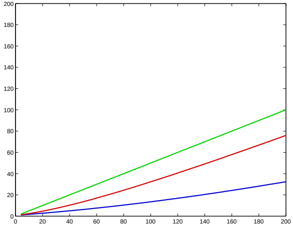
Two-Machine, Finite-Buffer Lines
Frequency and Average Inventory



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

Copyright ©2010 Stanley B. Gershwin.

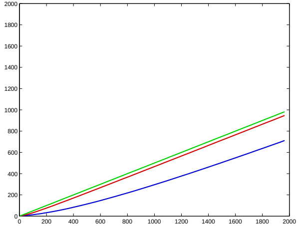
Two-Machine, Finite-Buffer Lines
Frequency and Average Inventory



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Frequency and Average Inventory



- M_1
  - $r_1 = .1, p_1 = .01$
- M_2
  - $r_2 = .1, p_2 = .01$ — Fast
  - $r_2 = .01, p_2 = .001$ — Slow
  - $r_2 = .001, p_2 = .0001$ — Very slow

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential processing time model

Exponential processing time: exponential processing, failure, and repair time; discrete state, continuous time; discrete material.

Assumptions are similar to deterministic processing time model, except:

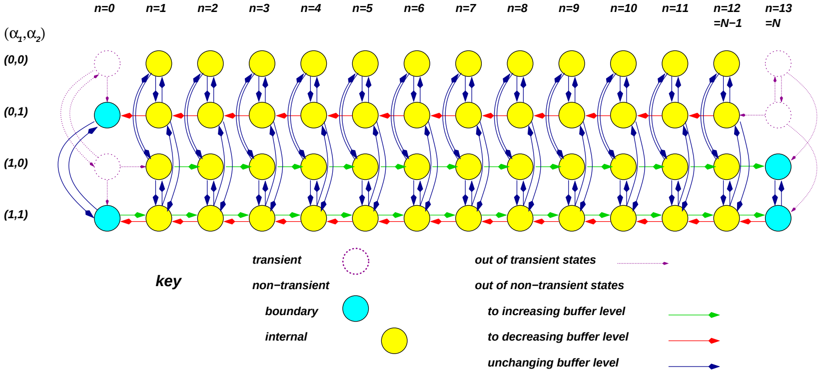
- $\mu_i \delta t$ = the probability that M_i completes an operation in $(t, t + \delta t)$;
- $p_i \delta t$ = the probability that M_i fails during an operation in $(t, t + \delta t)$;
- $r_i \delta t$ = the probability that M_i is repaired, while it is down, in $(t, t + \delta t)$;

We can assume that only one event occurs during $(t, t + \delta t)$.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential processing time model



 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

Performance measures for general exponential lines

The probability that Machine M_i is processing a workpiece is the *efficiency*:

$$E_i = \text{ prob } [\alpha_i = 1, n_{i-1} > 0, n_i < N_i].$$

The *production rate* (throughput rate) of Machine M_i, in parts per time unit, is

$$P_i = \mu_i E_i.$$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

Conservation of Flow

$$P = P_1 = P_2 = \ldots = P_k.$$

This should be proved from the model.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

Flow Rate-Idle Time Relationship

The *isolated efficiency* e_i of Machine M_i is, as usual,

$$e_i = \frac{r_i}{r_i + p_i}$$

and it represents the fraction of time that M_i is operational. The *isolated production rate* is

$$\rho_i = \mu_i e_i.$$

Two-Machine, Finite-Buffer Lines
Exponential processing time model

The flow rate-idle time relation is

$$E_i = e_i \text{ prob } [n_{i-1} > 0 \text{ and } n_i < N_i].$$

or

$$P = \rho_i \text{ prob } [n_{i-1} > 0 \text{ and } n_i < N_i].$$

This should also be proved from the model.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

Balance equations — steady state only

$\alpha_1 = a_2 = 0$:

$$\mathbf{p}(n,0,0)(r_1 + r_2) = \mathbf{p}(n,1,0)p_1 + \mathbf{p}(n,0,1)p_2, \quad 1 \leq n \leq N-1,$$

$$\mathbf{p}(0,0,0)(r_1 + r_2) = \mathbf{p}(0,1,0)p_1,$$

$$\mathbf{p}(N,0,0)(r_1 + r_2) = \mathbf{p}(N,0,1)p_2.$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential processing time model

$\boldsymbol{\alpha_1 = 0, \alpha_2 = 1:}$

$$\begin{aligned}\mathbf{p}(n,0,1)(r_1+\mu_2+p_2) &= \mathbf{p}(n,0,0)r_2 + \mathbf{p}(n,1,1)p_1 \\ &\quad + \mathbf{p}(n+1,0,1)\mu_2, 1 \leq n \leq N-1\end{aligned}$$

$$\mathbf{p}(0,0,1)r_1 = \mathbf{p}(0,0,0)r_2 + \mathbf{p}(0,1,1)p_1 + \mathbf{p}(1,0,1)\mu_2$$

$$\mathbf{p}(N,0,1)(r_1+\mu_2+p_2) = \mathbf{p}(N,0,0)r_2$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

$\boldsymbol{\alpha_1 = 1, \alpha_2 = 0:}$

$$\mathbf{p}(n,1,0)(p_1+\mu_1+r_2) = \mathbf{p}(n-1,1,0)\mu_1 + \mathbf{p}(n,0,0)r_1 + \mathbf{p}(n,1,1)p_2, 1 \le n \le N-1$$

$$\mathbf{p}(0,1,0)(p_1+\mu_1+r_2) = \mathbf{p}(0,0,0)r_1$$

$$\mathbf{p}(N,1,0)r_2 = \mathbf{p}(N-1,1,0\mu_1 + \mathbf{p}(N,0,0)r_1 + \mathbf{p}(N,1,1)p_2$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

$\alpha_1 = 1, \alpha_2 = 1:$

$$\begin{aligned} \mathbf{p}(n,1,1)(p_1+p_2+\mu_1+\mu_2) \quad &= \quad \mathbf{p}(n-1,1,1)\mu_1 + \mathbf{p}(n+1,1,1)\mu_2 \\ &+\mathbf{p}(n,1,0)r_2 + \mathbf{p}(n,0,1)r_1, \qquad 1 \le n \le N-1 \end{aligned}$$

$$\mathbf{p}(0,1,1)(p_1+\mu_1) = \mathbf{p}(1,1,1)\mu_2 + \mathbf{p}(0,1,0)r_2 + \mathbf{p}(0,0,1)r_1$$

$$\mathbf{p}(N,1,1)(p_2+\mu_2) = \mathbf{p}(N-1,1,1)\mu_1 + \mathbf{p}(N,1,0)r_2 + \mathbf{p}(N,0,1)r_1$$

 Copyright ©2010 Stanley B. Gershwin.

Performance measures

Efficiencies:

$$E_1 = \sum_{n=0}^{N-1} \sum_{\alpha_2=0}^{1} \mathbf{p}(n, 1, \alpha_2),$$

$$E_2 = \sum_{n=1}^{N} \sum_{\alpha_1=0}^{1} \mathbf{p}(n, \alpha_1, 1).$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential processing time model

Production rate:

$$P = \mu_1 E_1 = \mu_2 E_2.$$

Expected in-process inventory:

$$\bar{n} = \sum_{n=0}^{N} \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} n\mathbf{p}(n, \alpha_1, \alpha_2).$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Solution of balance equations

Assume

$$\mathbf{p}(n, \alpha_1, \alpha_2) = c X^n Y_1^{\alpha_1} Y_2^{\alpha_2}, \qquad 1 \leq n \leq N-1$$

where c, X, Y_1, Y_2 are parameters to be determined. Plugging this into the internal equations gives

$$p_1 Y_1 + p_2 Y_2 - r_1 - r_2 = 0$$

$$\mu_1 \left(\frac{1}{X} - 1 \right) - p_1 Y_1 + r_1 + \frac{r_1}{Y_1} - p_1 = 0$$

$$\mu_2 (X - 1) - p_2 Y_2 + \frac{r_2}{Y_2} + r_2 - p_2 = 0$$

 Copyright © 2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential processing time model

These equations can be reduced to one fourth-order polynomial (quartic) equation in one unknown. One solution is

$$Y_{11} = \frac{r_1}{p_1}$$
$$Y_{21} = \frac{r_2}{p_2}$$
$$X_1 = 1$$

This solution of the quartic equation has a zero coefficient in the expression for the probabilities of the internal states:

$$\mathbf{p}(n, \alpha_1, \alpha_2) = \sum_{j=1}^{4} c_j X_j^n Y_{1j}^{\alpha_1} Y_{2j}^{\alpha_2} \text{ for } n = 1, \ldots, N-1.$$

The other three solutions satisfy a cubic polynomial equation. *Compare with slide 85.* In general, there is no simple expression for them.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential processing time model

Just as for the deterministic processing time line,

- we obtain the coefficients c_1, c_2, c_3, c_4 from the boundary conditions and the normalization equation;
- we find $c_1 = 0$; *(What does this mean? Why is this true?)*
- we construct all the boundary probabilities. Some are 0.
- we use the probabilities to evaluate production rate, average buffer level, etc;
- we prove statements about conservation of flow, flow rate-idle time, limiting values of some quantities, etc.
- we draw graphs, and observe behavior which is qualitatively very similar to deterministic processing time line behavior (e.g., P vs. N, $\bar{n}$ vs N, etc.).

We also draw some new graphs (P vs. μ_i, $\bar{n}$ vs μ_i) and observe new behavior. This is discussed below with the discussion of continuous material lines.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Continuous Material model

Continuous material, or *fluid:* deterministic processing, exponential failure and repair time; mixed state, continuous time.; continuous material.

- $\mu_i \delta t$ = the amount of material that M_i processes, while it is up, in $(t, t + \delta t)$;
- $p_i \delta t$ = the probability that M_i fails, while it is up, in $(t, t + \delta t)$;
- $r_i \delta t$ = the probability that M_i is repaired, while it is down, in $(t, t + \delta t)$;

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Model assumptions, notation, terminology, and conventions

During time interval $(t, t + \delta t)$:

<u>When $0 < x < N$</u>

1. the change in x is $(\alpha_1 \mu_1 - \alpha_2 \mu_2)\delta t$
2. the probability of repair of Machine i, that is, the probability that $\alpha_i(t + \delta t) = 1$ given that $\alpha_i(t) = 0$, is $r_i \delta t$
3. the probability of failure of Machine i, that is, the probability that $\alpha_i(t + \delta t) = 0$ given that $\alpha_i(t) = 1$, is $p_i \delta t$.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

When $x = 0$

1. the change in x is $(\alpha_1\mu_1 - \alpha_2\mu_2)^+\delta t$
   (That is, when $x = 0$, it can only increase.)

2. the probability of repair is $r_i\delta t$

3. if Machine 1 is down, Machine 2 cannot fail. If Machine 1 is up, the probability of failure of Machine 2 is $p_2^b\delta t$, where

$$p_2^b = \frac{p_2\mu}{\mu_2}, \qquad \mu = \min(\mu_1, \mu_2)$$

   The probability of failure of Machine 1 is $p_1\delta t$.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

When $x = N$

1. the change in x is $(\alpha_1\mu_1 - \alpha_2\mu_2)^{-}\delta t$
2. the probability of repair is $r_i\delta t$
3. if Machine 2 is down, Machine 1 cannot fail. If Machine 2 is up, the probability of failure of Machine 1 is $p_1^b\delta t$, where

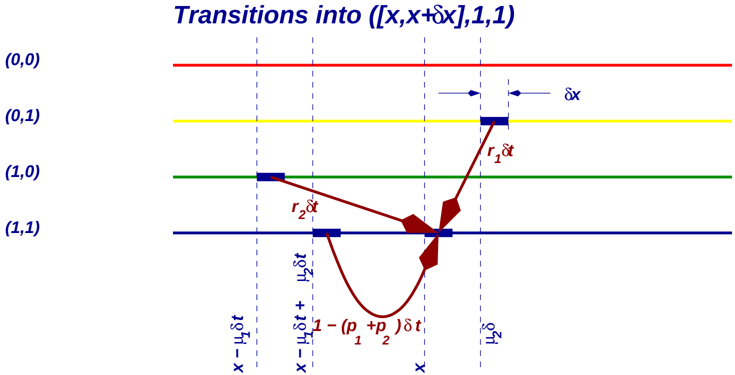
$$p_1^b = \frac{p_1\mu}{\mu_1}.$$

The probability of failure of Machine 2 is $p_2\delta t$.

Copyright ©2010 Stanley B. Gershwin.

Transition equations — internal

$f(x, \alpha_1, \alpha_2, t)\delta x + o(\delta x)$ is the probability of the buffer level being between x and $x + \delta x$ and the machines being in states α_1 and α_2 at time t.



Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Then

$$\begin{aligned} f(x,1,1,t+\delta t) \;&=\; (1-(p_1+p_2)\delta t) f(x-\mu_1\delta t+\mu_2\delta t,1,1,t) \\ &+ r_1\delta t f(x+\mu_2\delta t,0,1,t) + r_2\delta t f(x-\mu_1\delta t,1,0,t) \\ &+ o(\delta t) \end{aligned}$$

or

$$f(x,1,1,t+\delta t) =$$

$$(1-(p_1+p_2)\delta t)\left(f(x,1,1,t) + \frac{\partial f}{\partial x}(x,1,1,t)(-\mu_1\delta t+\mu_2\delta t)\right)$$

$$+ r_1\delta t f(x+\mu_2\delta t,0,1,t) + r_2\delta t f(x-\mu_1\delta t,1,0,t) + o(\delta t)$$

Two-Machine, Finite-Buffer Lines

or

$$f(x,1,1,t+\delta t) =$$

$$(1-(p_1+p_2)\delta t)\left(f(x,1,1,t)+\frac{\partial f}{\partial x}(x,1,1,t)(\mu_2-\mu_1)\delta t\right)$$

$$+r_1\delta t\left(f(x,0,1,t)+\frac{\partial f}{\partial x}(x,0,1,t)\mu_2\delta t\right)$$

$$+r_2\delta t\left(f(x,1,0,t)-\frac{\partial f}{\partial x}(x,1,0,t)\mu_1\delta t\right)+o(\delta t)$$

Copyright ©2010 Stanley B. Gershwin.

or

$$f(x,1,1,t+\delta t) =$$

$$f(x,1,1,t) - (p_1+p_2) f(x,1,1,t)\delta t + (\mu_2 - \mu_1)\frac{\partial f}{\partial x}(x,1,1,t)\delta t$$

$$+ r_1 f(x,0,1,t)\delta t + r_2 f(x,1,0,t)\delta t$$

or, finally,

$$\boxed{\begin{array}{c}\dfrac{\partial f}{\partial t}(x,1,1) = -(p_1+p_2) f(x,1,1) + (\mu_2-\mu_1)\dfrac{\partial f}{\partial x}(x,1,1) \\ + r_1 f(x,0,1) + r_2 f(x,1,0)\end{array}}$$

Similarly,

$$\frac{\partial f}{\partial t}(x,0,0) = -(r_1 + r_2)f(x,0,0) + p_1 f(x,1,0) + p_2 f(x,0,1)$$

$$\frac{\partial f}{\partial t}(x,0,1) = \mu_2 \frac{\partial f}{\partial x}(x,0,1) - (r_1 + p_2)f(x,0,1) + p_1 f(x,1,1) + r_2 f(x,0,0)$$

$$\frac{\partial f}{\partial t}(x,1,0) = -\mu_1 \frac{\partial f}{\partial x}(x,1,0) - (p_1 + r_2)f(x,1,0) + p_2 f(x,1,1) + r_1 f(x,0,0)$$

Two-Machine, Finite-Buffer Lines

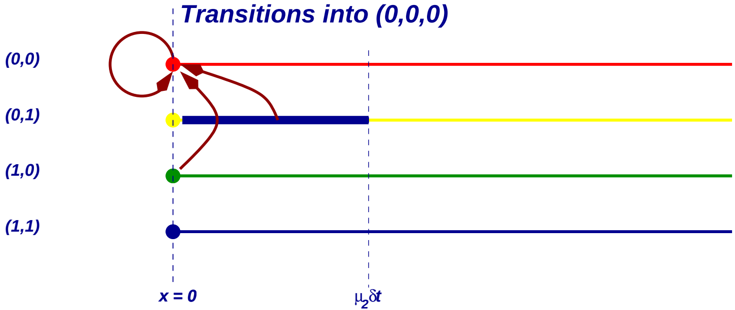
Transition equations — boundary

$\mathbf{p}(x, \alpha_1, \alpha_2, t)$ is the probability of the buffer level being x (where $x = 0$ or N) and the machines being in states α_1 and α_2 at time t.

Boundary equations describe transitions from boundary states to boundary states; from boundary states to interior states; and from interior states to boundary states.

Boundary equations are relationships among $\mathbf{p}(x, \alpha_1, \alpha_2, t)$ and $f(x, \alpha_1, \alpha_2, t)$ and their derivatives for $x = 0$ or $x = N$.

 Copyright ©2010 Stanley B. Gershwin.

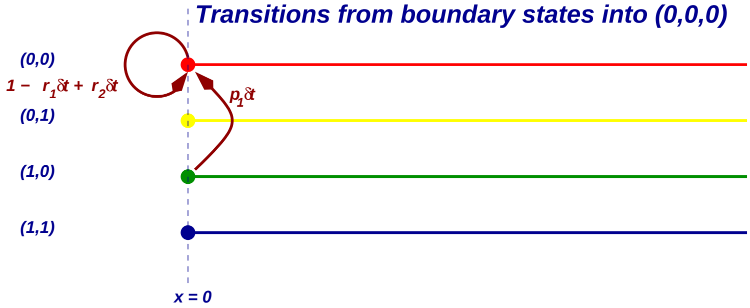


We must construct an equation of the form

$$\mathbf{p}(0,0,0,t+\delta t) = \mathbf{p}(0,0,0,t) + A\delta t + o(\delta t)$$

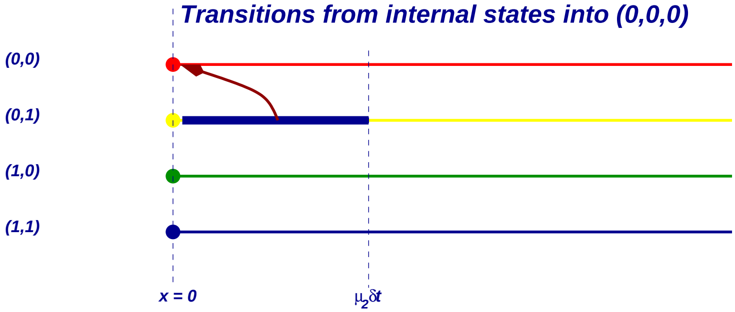
Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines



The system can go from (0,0,0) to (0,0,0) if there is no repair. It can go from (0,1,0) if the first machine does not fail.

It *cannot* go from (0,0,1) to (0,0,0) because the second machine is starved and cannot fail. To go from (0,1,1) to (0,0,0) require two simultaneous failures, which has a probability on the order of δt^2.

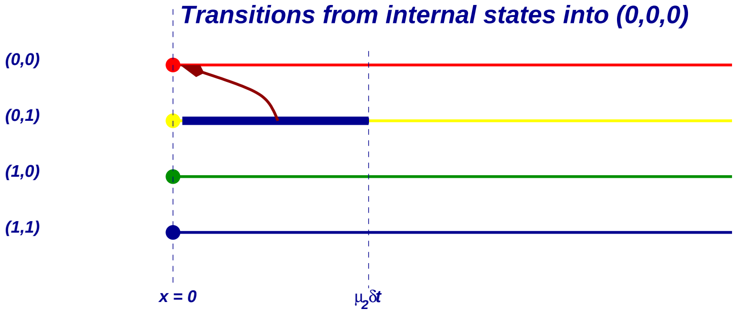


To go from $(x, \alpha_1, \alpha_2), x > 0$ to (0,0,0), we must have

$$0 < x < \alpha_2 \mu_2 \delta t - \alpha_1 \mu_1 \delta t$$

For example, if $\alpha_1 = 0$ and $\alpha_2 = 1$, we are considering transitions from $(x, 0, 1)$ to (0,0,0) where $0 < x < \mu_2 \delta t$.

 Copyright ©2010 Stanley B. Gershwin.



But

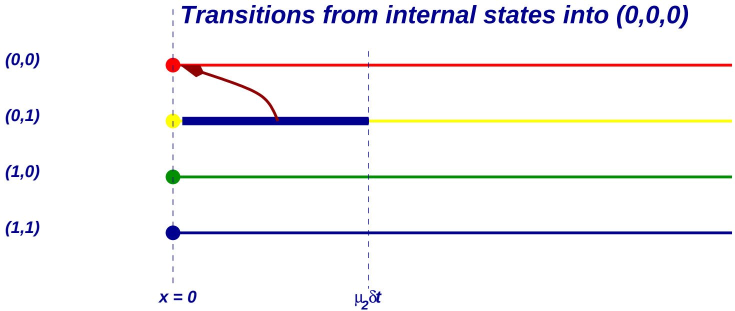
$$\text{prob}\left([0 < x < \mu_2 \delta t], 0, 1\right) = f(x,0,1)\mu_2\delta t + o(\delta t) = f(0,0,1)\mu_2\delta t + o(\delta t)$$

and the transition probability from (0,1) to (0,0) is

$$(1 - r_1\delta t)p_2\delta t + o(\delta t) = p_2\delta t + o(\delta t).$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines



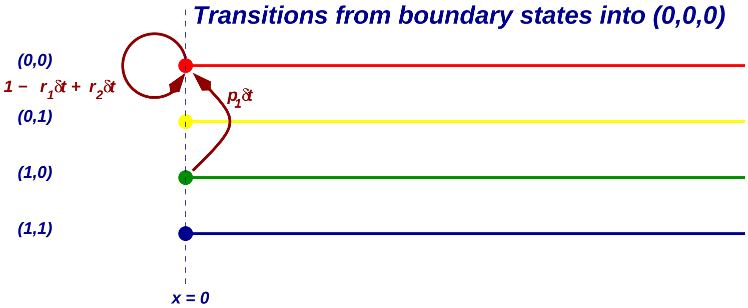
Therefore, the probability of going from $([0 < x < \mu_2 \delta t], 0, 1)$ to (0,0,0) is

$$f(x, 0, 1)\mu_2 p_2 \delta t^2 + (\delta t) o(\delta t) = o(\delta t)$$

For other transitions from $(x, \alpha_1, \alpha_2), x > 0$ to (0,0,0), the probabilities are similar or smaller.

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines



Therefore

$$\mathbf{p}(0,0,0,t+\delta t) = (1 - r_1\delta t - r_2\delta t)\mathbf{p}(0,0,0,t) + \mathbf{p}(0,1,0,t)p_1\delta t$$

or

$$\frac{d}{dt}\mathbf{p}(0,0,0) = -(r_1 + r_2)\mathbf{p}(0,0,0) + p_1\mathbf{p}(0,1,0)$$

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Consider state (0,1,0). As soon as the system enters this state, it leaves. This is because x must immediately increase. Therefore

$$\mathbf{p}(0,1,0) = 0$$

even if the system is not in steady state . Therefore

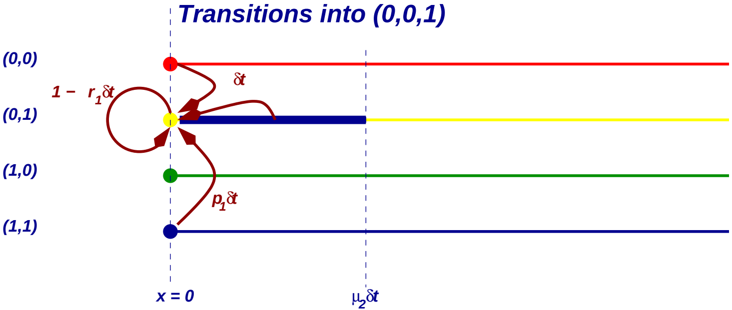
$$\frac{d}{dt}\mathbf{p}(0,0,0) = -(r_1 + r_2)\mathbf{p}(0,0,0)$$

In steady state,

$$\mathbf{p}(0,0,0) = 0$$

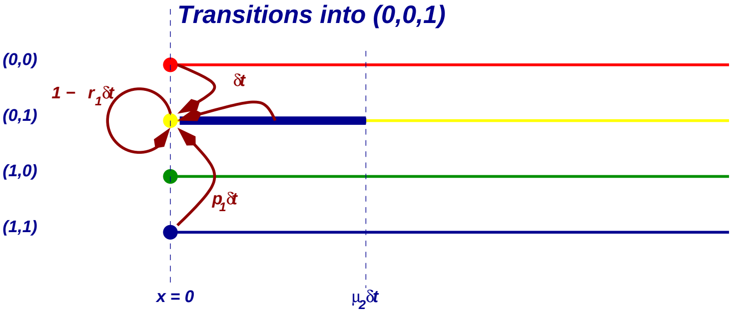
 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines



$$\mathbf{p}(0,0,1,t+\delta t) = r_2\delta t\mathbf{p}(0,0,0,t) + (1 - r_1\delta t)\mathbf{p}(0,0,1,t) + p_1\delta t\mathbf{p}(0,1,1,t) + \int_0^{\mu_2\delta t} f(x,0,1,t)dx$$

 Copyright © 2010 Stanley B. Gershwin.

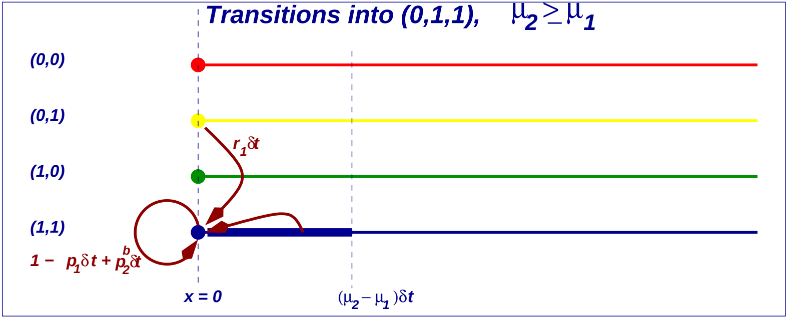


or,

$$\frac{d}{dt}\mathbf{p}(0,0,1) = r_2\mathbf{p}(0,0,0) - r_1\mathbf{p}(0,0,1) + p_1\mathbf{p}(0,1,1) + \mu_2 f(0,0,1).$$

 Copyright ©2010 Stanley B. Gershwin.

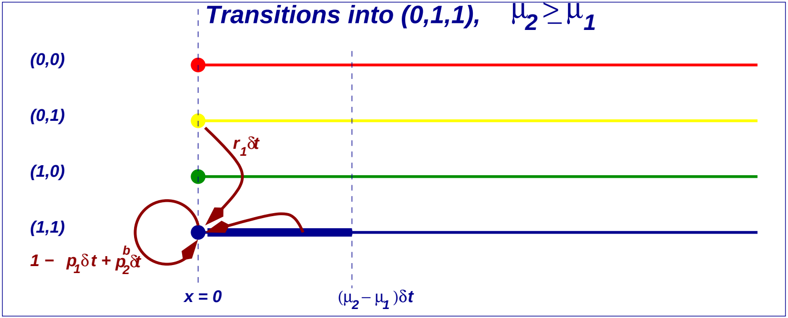
Two-Machine, Finite-Buffer Lines



$$\mathbf{p}(0,1,1,t+\delta t) = (1-(p_1+p_2^b)\delta t)\mathbf{p}(0,1,1,t) + r_1\delta t\mathbf{p}(0,0,1,t) + \int_0^{(\mu_2-\mu_1)\delta t} f(x,1,1,t)dx,$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines



$$\frac{d}{dt}\mathbf{p}(0,1,1) = -(p_1 + p_2^b)\mathbf{p}(0,1,1) + r_1\mathbf{p}(0,0,1)$$
$$+(\mu_2 - \mu_1)f(0,1,1), \text{ if } \mu_1 \le \mu_2.$$

 Copyright ©2010 Stanley B. Gershwin.

Transitions into (0,1,1), $\mu_2 \leq \mu_1$

If $x(t) = 0$, the transition from any $(\alpha_1(t), \alpha_2(t))$ to $(\alpha_1(t+\delta t), \alpha_2(t+\delta t)) = (1,1)$ would cause x to increase immediately. Therefore

$$\mathbf{p}(0,1,1) = 0$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

To come:

- Other boundary equations
- Normalization

$$\sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \left[\int_0^N f(x, \alpha_1, \alpha_2) dx + \mathbf{p}(0, \alpha_1, \alpha_2) + \mathbf{p}(N, \alpha_1, \alpha_2) \right] = 1.$$

- Production rate

$$P_2 = \mu_2 \left[\int_0^N (f(x, 0, 1) + f(x, 1, 1)) dx + p(N, 1, 1) \right] + \mu_1 \mathbf{p}(0, 1, 1).$$

$$= P_1 = \mu_1 \left[\int_0^N (f(x, 1, 0) + f(x, 1, 1)) dx + \mathbf{p}(0, 1, 1) \right] + \mu_2 \mathbf{p}(N, 1, 1).$$

- Average in-process inventory

$$\bar{x} = \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \left[\int_0^N x f(x, \alpha_1, \alpha_2) dx + N \mathbf{p}(N, \alpha_1, \alpha_2) \right].$$

 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Also to come:

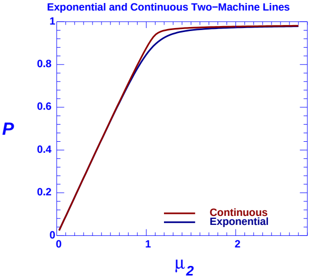
- Identities (in steady state)
  - Conservation of flow; Blocking, Starvation, and Production Rate; Repair frequency equals failure frequency; Flow Rate-Idle Time; Limits
  - Solution technique
    - Internal solution; transient states;

$$f(x, \alpha_1, \alpha_2) = Ce^{\lambda x} Y_1^{\alpha_1} Y_2^{\alpha_2}$$

Cases ($\mu_1 < \mu_2, \mu_1 = \mu_2, \mu_1 > \mu_2$); boundary probabilities

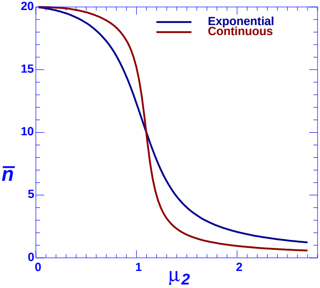
 Copyright ©2010 Stanley B. Gershwin.

- $r_1 = 0.09$, $p_1 = 0.01$, $\mu_1 = 1.1$
- $r_2 = 0.08$, $p_1 = 0.009$
- $N = 20$
- *Explain the shapes of the graphs.*



 Copyright ©2010 Stanley B. Gershwin.

- *Explain the shapes of the graphs.*



 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines
Exponential and continuous line performance

The no-variability limit:

Consider a new continuous-material two-machine line. It is has parameters $\mu'_1, r'_1, p'_1, \mu'_2, r'_2, p'_2, N'$. Assume it is *perfectly reliable* and its machines have the same *isolated production rates* as those of the first continuous-material two-machine line. It also has the same buffer size.

Its parameters are therefore given by

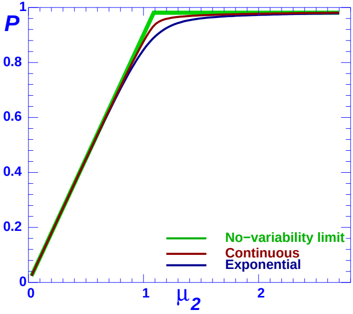
$$\begin{array}{llll} \mu'_1 = \rho_1; & r'_1 \text{ unspecified}; & p'_1 = 0; & N' = N \\ \mu'_2 = \rho_2; & r'_2 \text{ unspecified}; & p'_2 = 0 & \end{array}$$

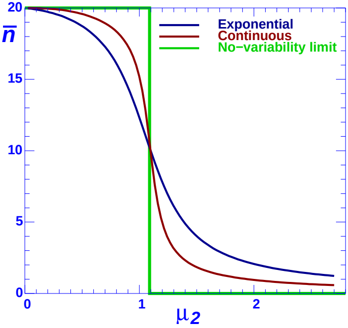
where

$$\rho_i = \mu_i \frac{r_i}{r_i + p_i}$$

Two-Machine, Finite-Buffer Lines

Exponential and continuous line performance

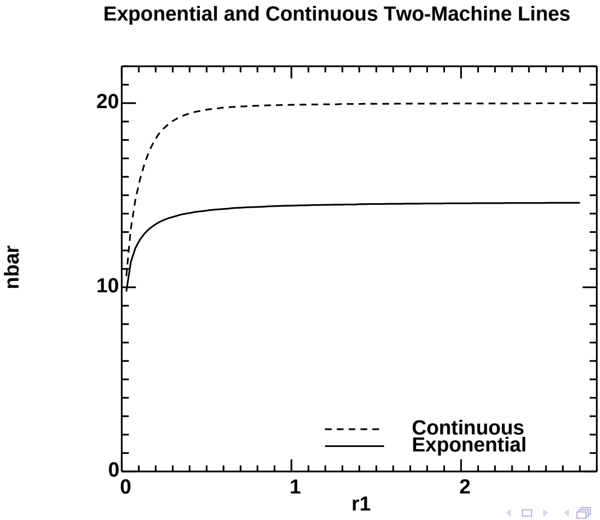




 Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Exponential and continuous line performance



 Copyright ©2010 Stanley B. Gershwin.

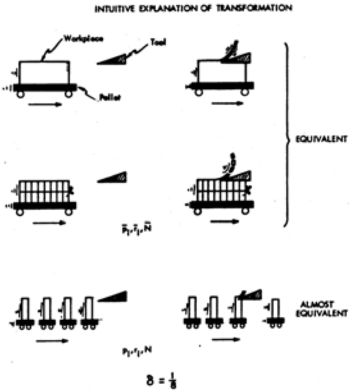
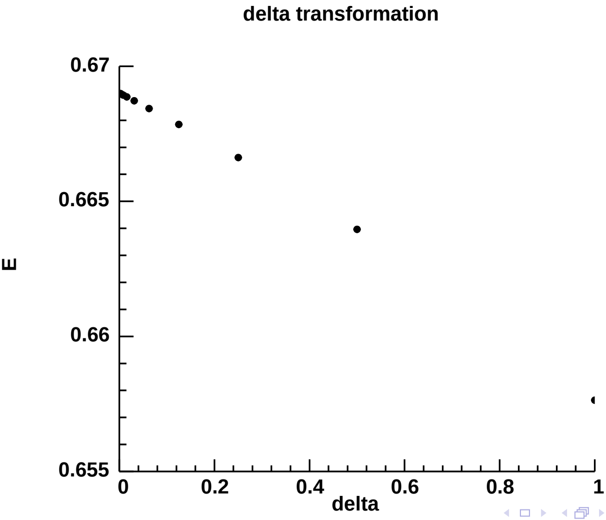


Figure 6.8 in Schick, Irvin C. "Analysis of a multistage transfer line with unreliable components and interstage buffer storages with applications to chemical engineering problems." Master's thesis, MIT, 1978.

Copyright ©2010 Stanley B. Gershwin.

Two-Machine, Finite-Buffer Lines

Continuous material and Deterministic Processing Time Lines



 Copyright ©2010 Stanley B. Gershwin.

MIT OpenCourseWare
http://ocw.mit.edu

2.852 Manufacturing Systems Analysis
Spring 2010

For information about citing these materials or our Terms of Use,visit: http://ocw.mit.edu/terms.