MAS.160 / MAS.510 / MAS.511 Signals, Systems and Information for Media Technology
Fall 2007

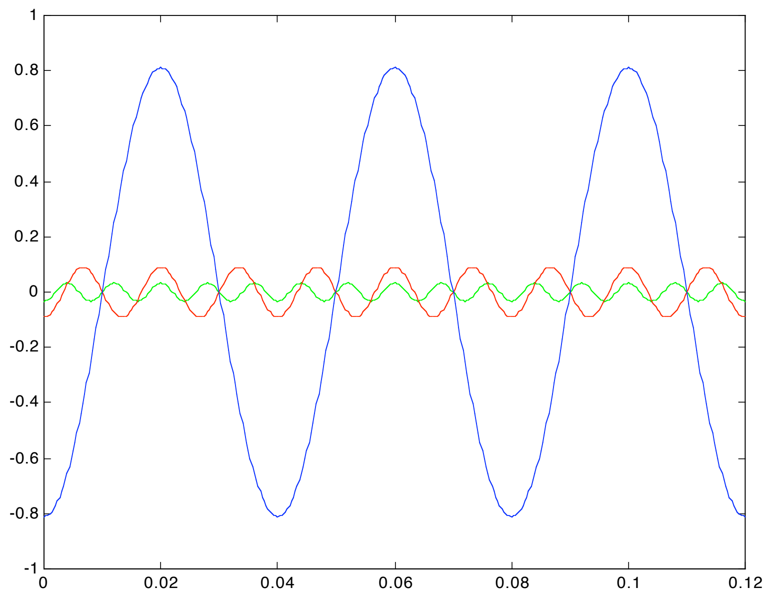
Composite signals (waveform synthesis)

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_k = \begin{cases} \dfrac{8}{\pi^2 k^2} e^{j\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

k=5

$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi)$$



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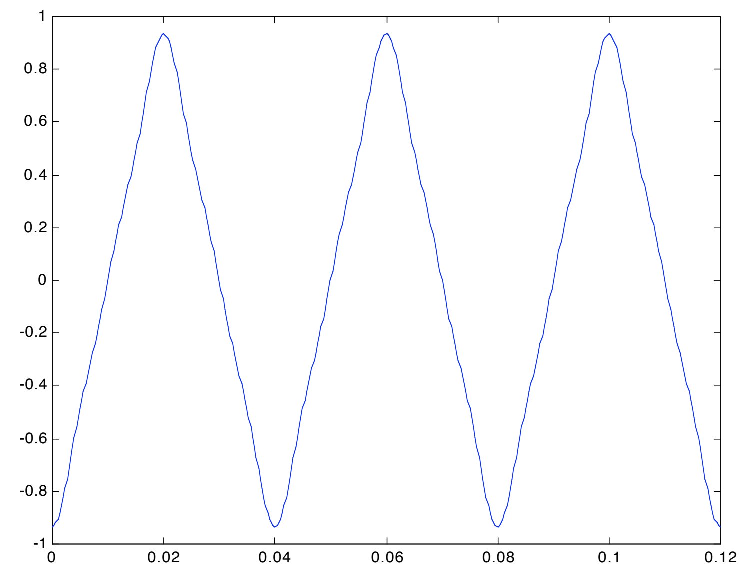
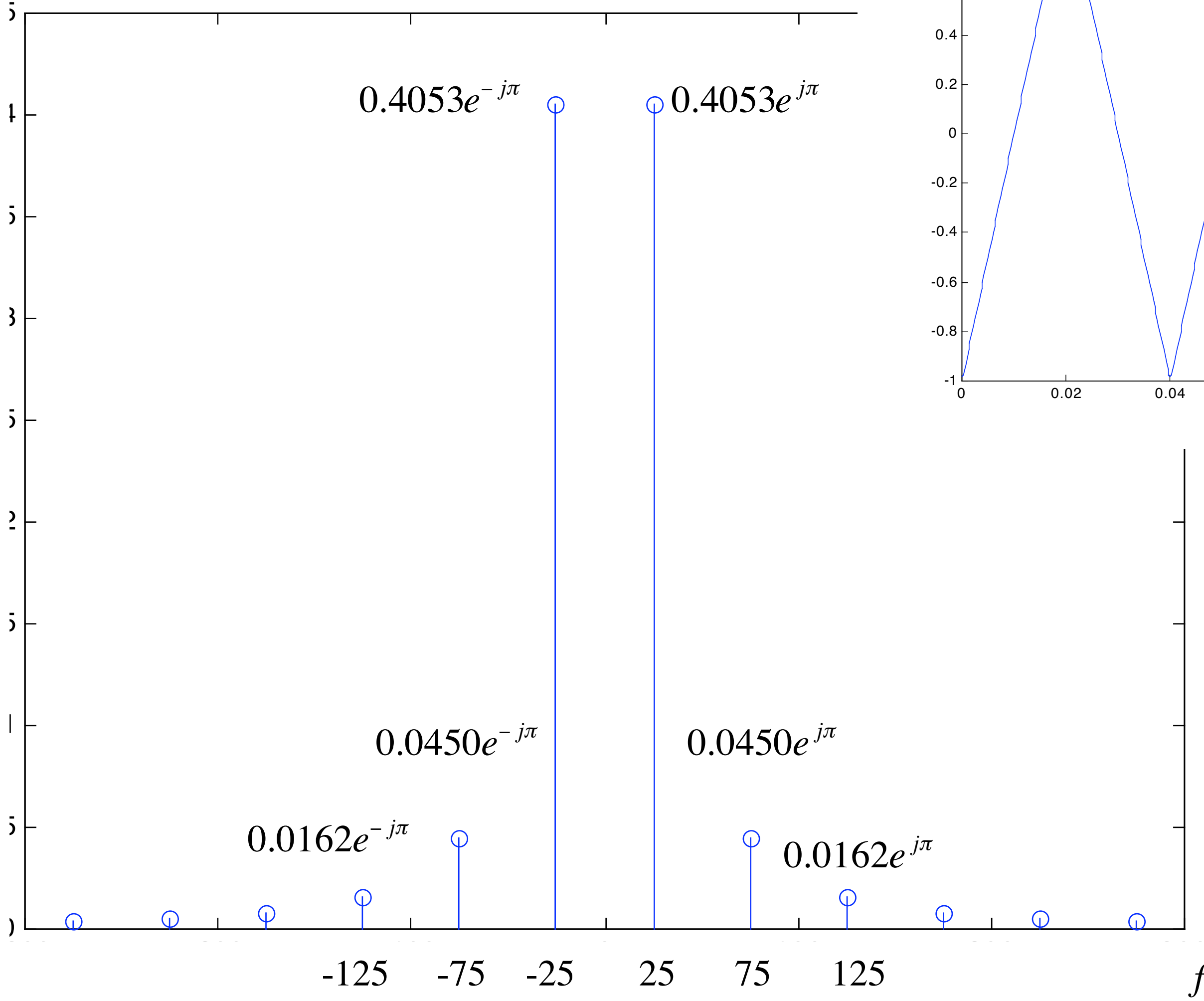
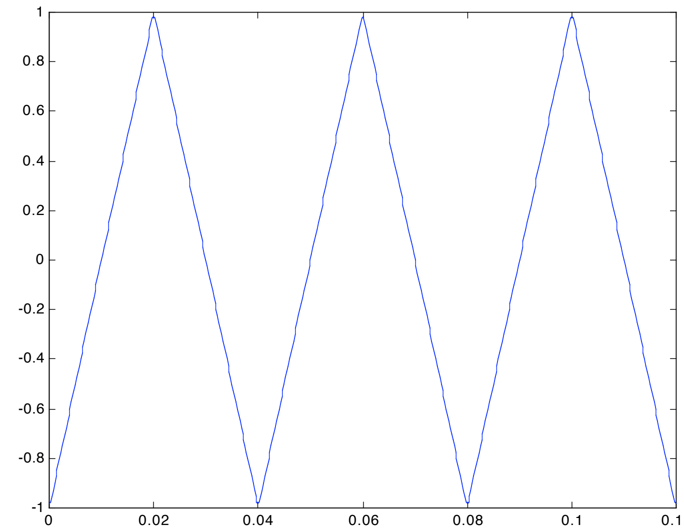


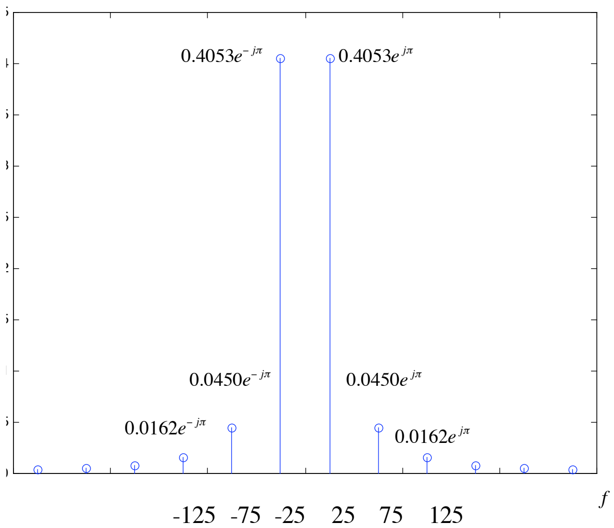
spectrum

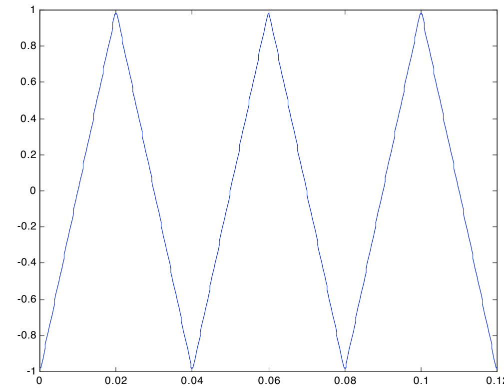




$$x(t) = 0.8105\cos(2\pi 25t + \pi) + 0.0901\cos(2\pi 75t + \pi) + 0.0324\cos(2\pi 125t + \pi) + \ldots$$

spectrum



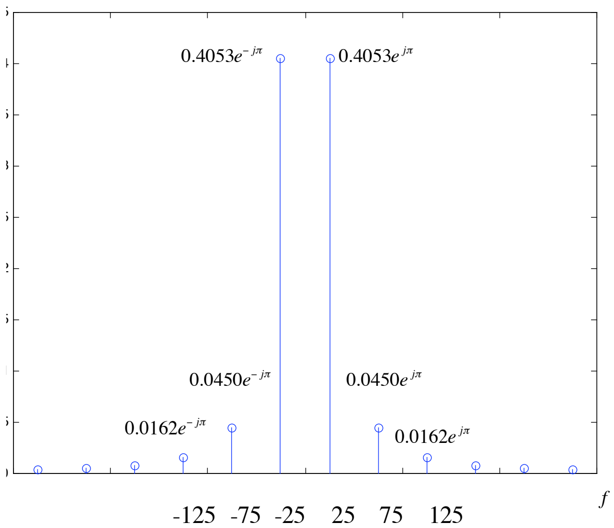


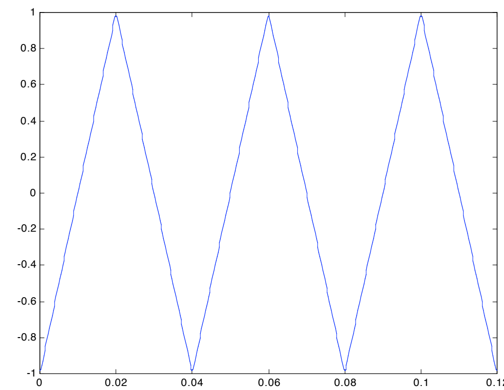
$$x(t) = 0.8105\cos\left(2\pi 25t + \pi\right) + 0.0901\cos\left(2\pi 75t + \pi\right) + 0.0324\cos\left(2\pi 125t + \pi\right) + \ldots$$

Complex conjugate form

$$x(t) = 0.8105\,\frac{e^{j\left(2\pi 25t + \pi\right)} + e^{-j\left(2\pi 25t + \pi\right)}}{2} + 0.0901\,\frac{e^{j\left(2\pi 75t + \pi\right)} + e^{-j\left(2\pi 75t + \pi\right)}}{2} + 0.0324\,\frac{e^{j\left(2\pi 125t + \pi\right)} + e^{-j\left(2\pi 125t + \pi\right)}}{2} + \ldots$$

spectrum



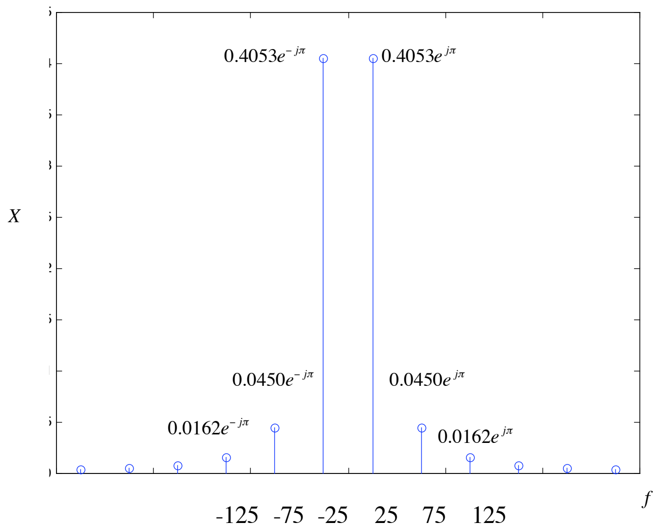


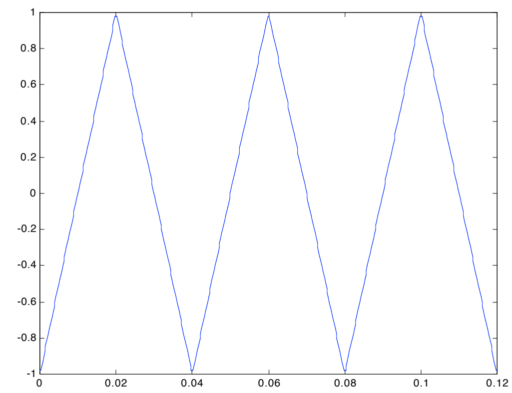
Complex conjugate form

$$x(t) = 0.8105\,\frac{e^{j(2\pi 25t+\pi)} + e^{-j(2\pi 25t+\pi)}}{2} + 0.0901\,\frac{e^{j(2\pi 75t+\pi)} + e^{-j(2\pi 75t+\pi)}}{2} + 0.0324\,\frac{e^{j(2\pi 125t+\pi)} + e^{-j(2\pi 125t+\pi)}}{2} + \ldots$$

$$x(t) = 0.4053e^{j(2\pi 25t+\pi)} + 0.4053e^{-j(2\pi 25t+\pi)} + 0.0450e^{j(2\pi 75t+\pi)} + 0.0450e^{-j(2\pi 75t+\pi)}$$
$$+ 0.0162e^{j(2\pi 125t+\pi)} + 0.0162e^{-j(2\pi 125t+\pi)} + \ldots$$

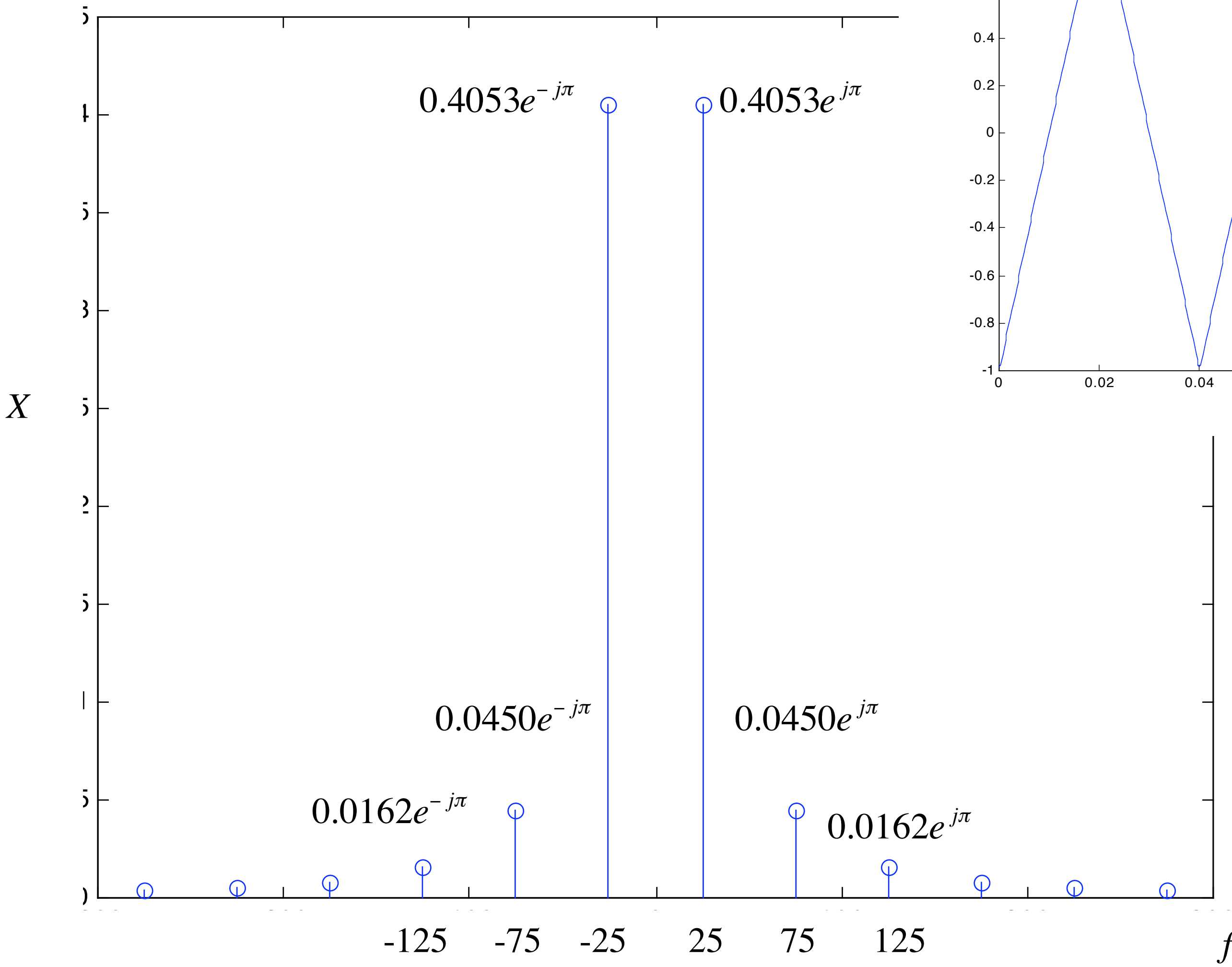
spectrum

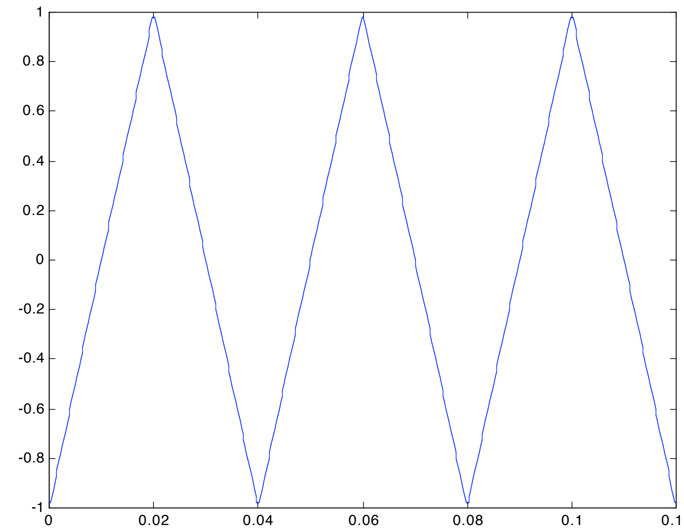




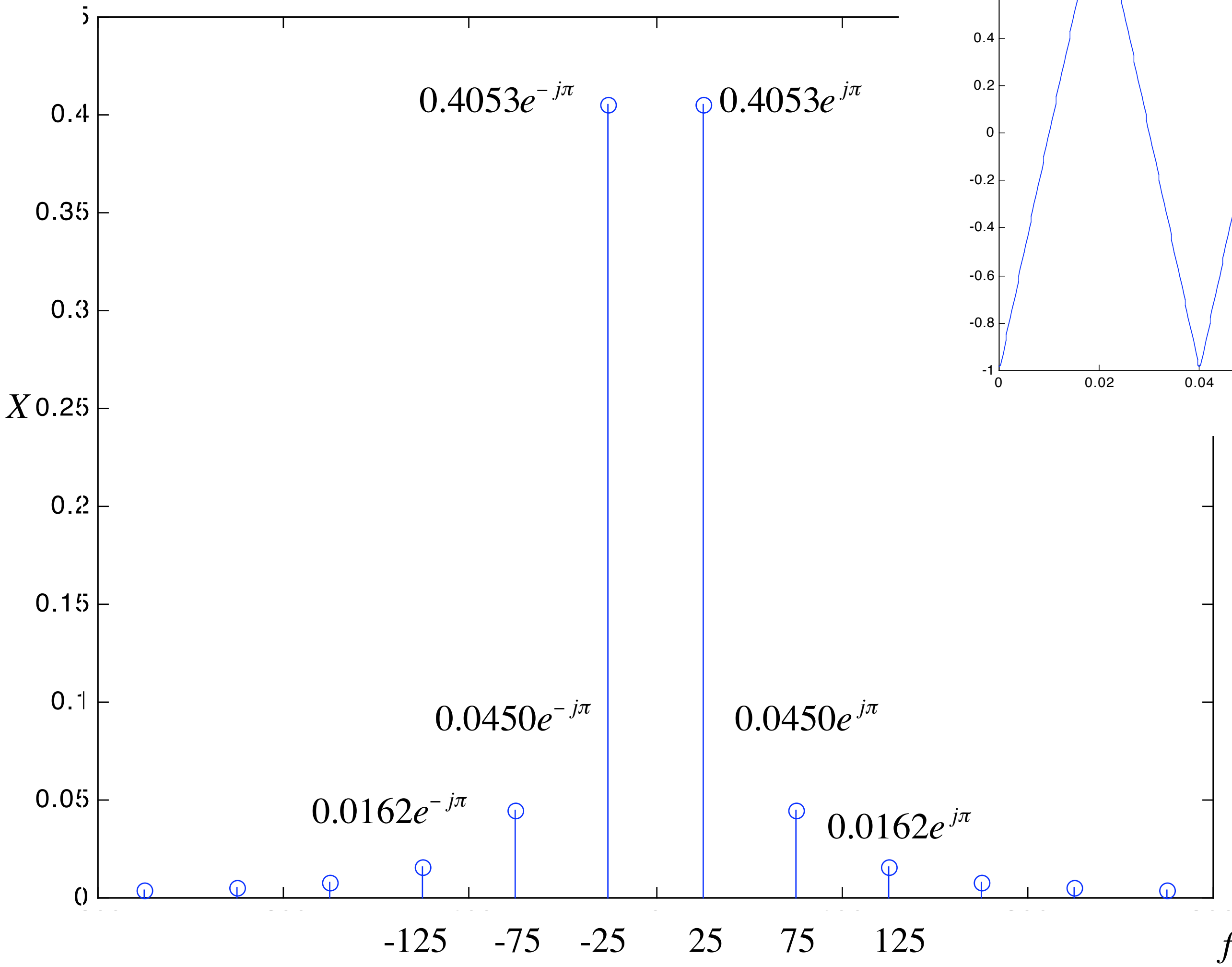
$$x(t) = 0.4053e^{j(2\pi 25t+\pi)} + 0.4053e^{-j(2\pi 25t+\pi)} + 0.0450e^{j(2\pi 75t+\pi)} + 0.0450e^{-j(2\pi 75t+\pi)}$$
$$+ 0.0162e^{j(2\pi 125t+\pi)} + 0.0162e^{-j(2\pi 125t+\pi)} + \ldots$$

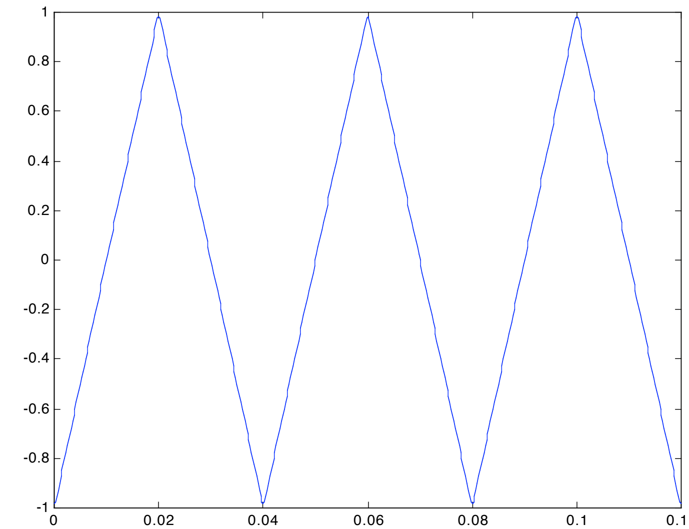
$$x(t) = 0.4053e^{j\pi}e^{j(2\pi 25t)} + 0.4053e^{-j\pi}e^{-j(2\pi 25t)} + 0.0450e^{j\pi}e^{j(2\pi 75t)} + 0.0450e^{-j\pi}e^{-j(2\pi 75t)}$$
$$+ 0.0162e^{j\pi}e^{j(2\pi 125t)} + 0.0162e^{-j\pi}e^{-j(2\pi 125t)} + \ldots$$





$$x(t) = 0.0162e^{-j\pi}e^{-j(2\pi 125t)} + 0.0450e^{-j\pi}e^{-j(2\pi 75t)} + 0.4053e^{-j\pi}e^{-j(2\pi 25t)}$$
$$+ 0.4053e^{j\pi}e^{j(2\pi 25t)} + 0.0450e^{j\pi}e^{j(2\pi 75t)} + 0.0162e^{j\pi}e^{j(2\pi 125t)} + \ldots$$





"two sided Fourier Series"

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = \mathrm{Re}\left\{\sum_{n=0}^{\infty} X_k e^{j2\pi k f_0 t}\right\} = \sum_{k=-\infty}^{\infty} Z_k e^{j2\pi k f_0 t}$$

Fourier Series

For a given signal, how do we find $X_k = A_k e^{j\phi_k}$ for each k in the Fourier Series ?

Fourier Analysis

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

where

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

f_0:fundamental frequency

$T_0 = 1/f_0$

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt$$

Fourier Series: Sawtooth

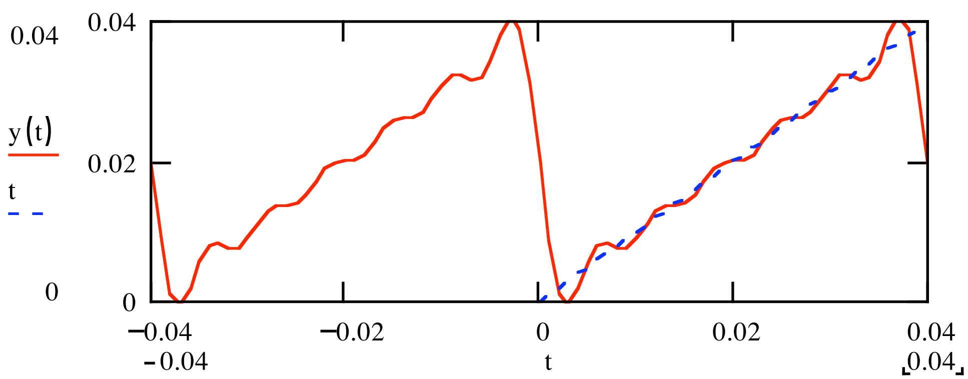
$$x(t) = t \quad 0 \le t < T_0$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$X_0 = \frac{T_0}{2} \qquad X_k = \frac{T_0}{\pi k} e^{j\frac{\pi}{2}}$$

$$x(t) = \frac{T_0}{2} + \sum_{k=1}^{\infty} \frac{T_0}{\pi k} \cos(2\pi k f_0 t + \tfrac{\pi}{2})$$

$$x(t) = \frac{T_0}{2} + \frac{T_0}{\pi} \cos(2\pi f_0 t + \tfrac{\pi}{2}) + \frac{T_0}{2\pi} \cos(2\pi 2 f_0 t + \tfrac{\pi}{2}) + \ldots$$



Defined between 0<t<0.04

Periodic with period 0.04

Fourier Series:Square Wave

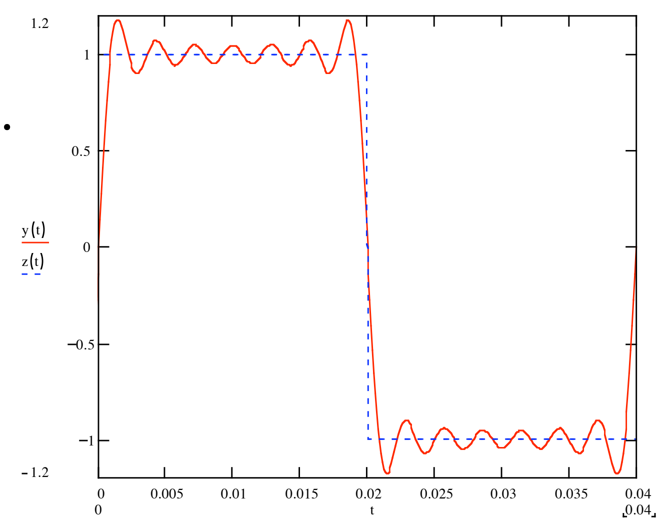
$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

$$X_0 = 0 \qquad X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases} \longrightarrow X_k = \begin{cases} \dfrac{4}{k\pi}e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$

$$x(t) = \tfrac{4}{\pi}\sin\left(2\pi f_0 t\right) + \tfrac{4}{3\pi}\sin\left(2\pi 3 f_0 t\right) + \ldots$$

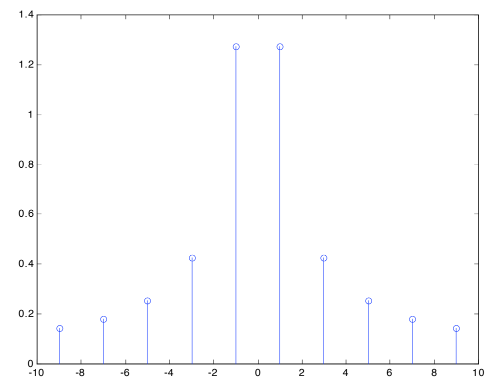


Fourier Series:Square Wave Spectrum

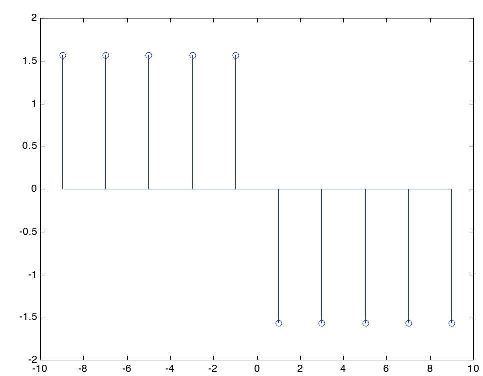
$$X_0 = 0 \qquad X_k = \begin{cases} -j\dfrac{4}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases} \longrightarrow X_k = \begin{cases} \dfrac{4}{k\pi}e^{-j\frac{\pi}{2}} & k \quad odd(1,3,5\ldots) \\ 0 & k \quad even(2,4,6\ldots) \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi k f_0 t + \phi_k\right) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \tfrac{4}{\pi}\sin\left(2\pi f_0 t\right) + \tfrac{4}{3\pi}\sin\left(2\pi 3 f_0 t\right) + \ldots$$



amp



phase

Complex conjugate form

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_k \left(\frac{e^{j2\pi k f_0 t} + e^{-j2\pi k f_0 t}}{2}\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \left(\frac{e^{j2\pi k f_0 t}}{2}\right) = \sum_{k=-\infty}^{\infty} Z_k e^{j2\pi k f_0 t}$$

"two sided Fourier Series"

$$Z_0 = 0 \quad Z_k = \begin{cases} -j\dfrac{2}{k\pi} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4 \ldots \end{cases}$$

$$Z_k = \frac{X_k}{2} = \begin{cases} \dfrac{2}{|k|\pi}e^{-\frac{\pi}{2}k} & k = \pm 1, \pm 3 \ldots \\ 0 & k = \pm 2, \pm 4 \ldots \end{cases}$$

Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

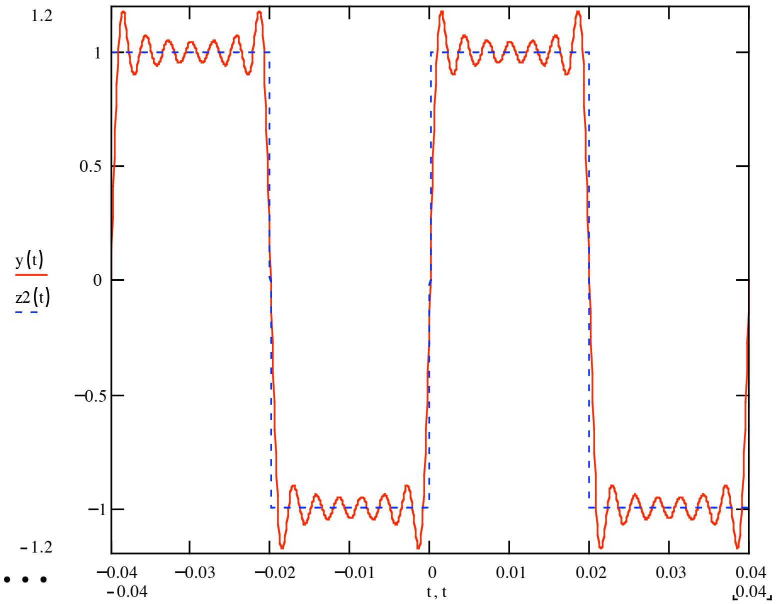
Even functions $f(-x) = f(x)$ consist only of sums of cosines ($\phi = 0$)

Ex. Square wave

$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

odd function

$$X_0 = 0 \qquad X_k = \begin{cases} \dfrac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$



$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$

$$x(t) = \tfrac{4}{\pi}\sin\left(2\pi f_0 t\right) + \tfrac{4}{3\pi}\sin\left(2\pi 3 f_0 t\right) + \ldots$$

Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

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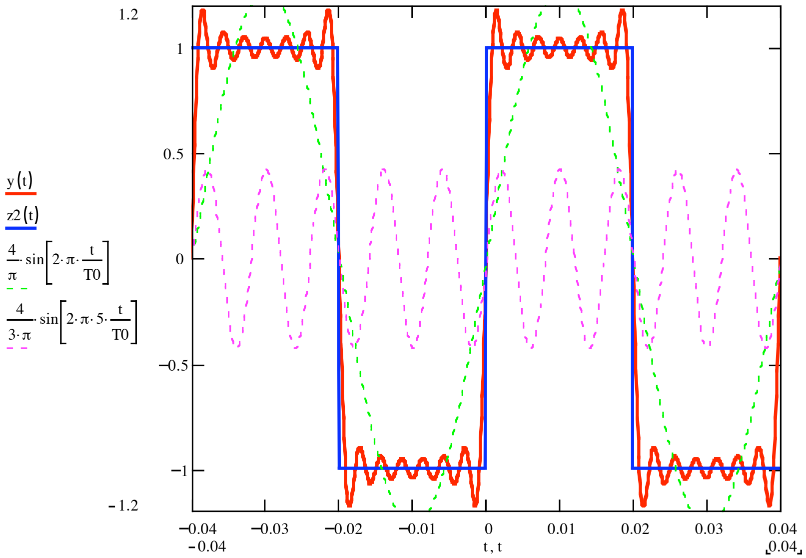

$$X_0 = 0 \qquad X_k = \begin{cases} \dfrac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$

$$x(t) = \tfrac{4}{\pi}\sin\left(2\pi f_0 t\right) + \tfrac{4}{3\pi}\sin\left(2\pi 3 f_0 t\right) + \ldots$$

Properties of Fourier Series

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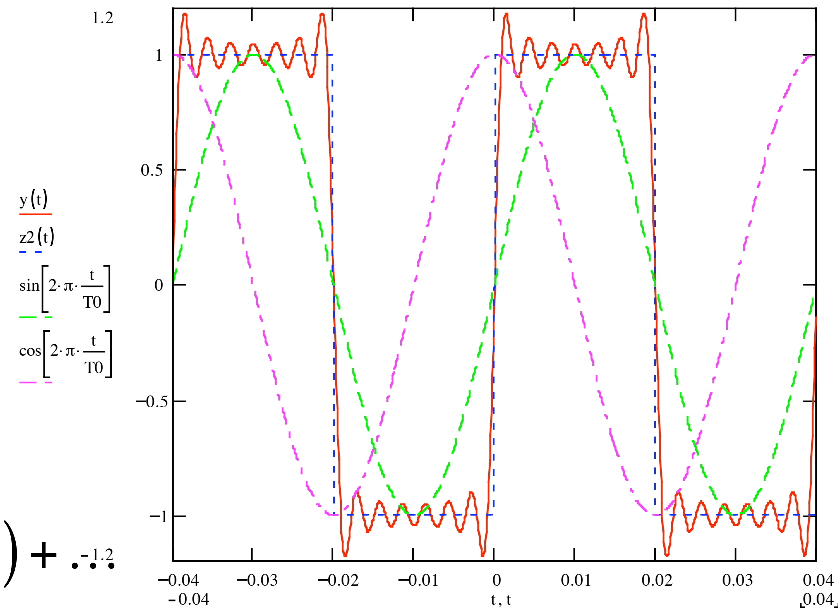
$$x(t) = \begin{cases} 1 & 0 \le t < T_0/2 \\ -1 & T_0/2 \le t < T_0 \end{cases}$$

odd function

$$X_0 = 0$$

$$X_k = \begin{cases} \frac{4}{k\pi} e^{-j\frac{\pi}{2}} & k \quad odd \\ 0 & k \quad even \end{cases}$$



$$x(t) = \tfrac{4}{\pi}\cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right) + \tfrac{4}{3\pi}\cos\left(2\pi 3 f_0 t - \tfrac{\pi}{2}\right) + \ldots$$

$$x(t) = \tfrac{4}{\pi}\sin\left(2\pi f_0 t\right) + \tfrac{4}{3\pi}\sin\left(2\pi 3 f_0 t\right) + \ldots$$

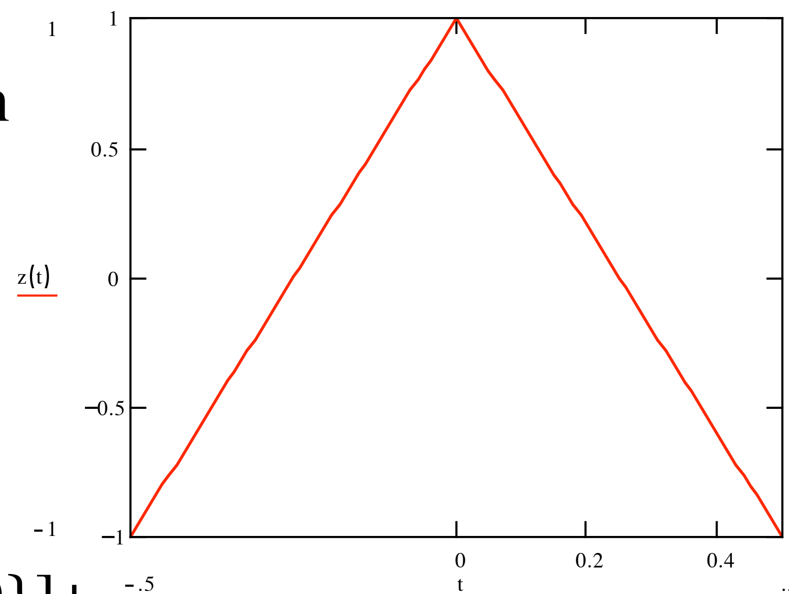
Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

Even functions $f(-x) = f(x)$ consist only of sums of cosines ($\phi = 0$)

Ex.Triangle wave

$$x(t) = \begin{cases} 4t + T_0 & -T_0/2 \le t < 0 \\ -4t + T_0 & 0 \le t < T_0/2 \end{cases}$$ even function



$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$X_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0}(-4t - T_0)dt + \frac{1}{T_0}\int_0^{T_0/2}(4t - T_0)dt$$

In[1]:= 1/T*Integrate[-4*t-T,{t,-T/2,0}]+ 1/T*Integrate[4*t-T,{t,0,T/2}]

Out[1]= 0

$X_0 = 0$

Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

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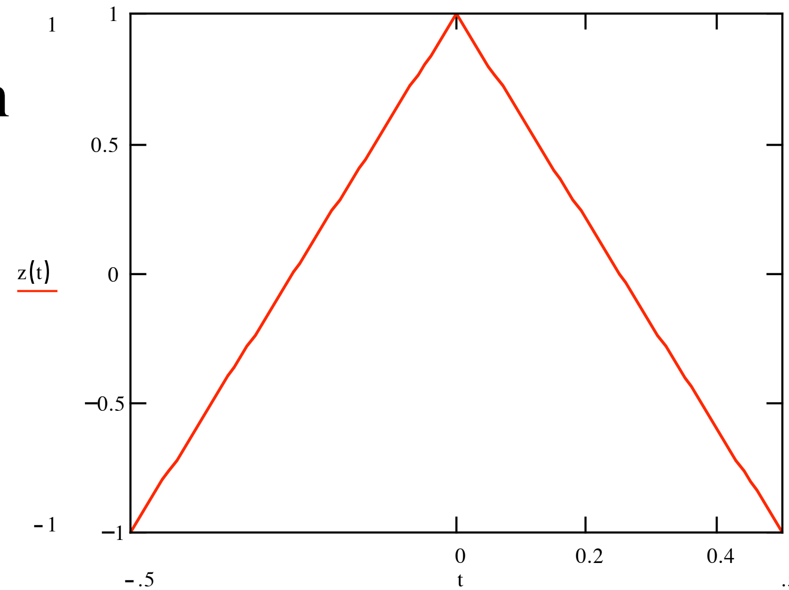


$$X_k = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi kt/T_0} dt$$

$$X_k = \frac{2}{T_0} \int_{0}^{T_0/2} (-4t + T_0) e^{-j2\pi kt/T_0} dt + \frac{2}{T_0} \int_{-T_0/2}^{0} (4t + T_0) e^{-j2\pi kt/T_0} dt$$

Ex. $$X_k = \frac{2}{T_0}\int_0^{T_0/2}\left(-4t + T_0\right)e^{-j2\pi kt/T_0}\,dt + \frac{2}{T_0}\int_{-T_0/2}^{0}\left(4t + T_0\right)e^{-j2\pi kt/T_0}\,dt$$

```
In[3]:=  2/T*Integrate[(T+4*t)*Exp[-I*2*Pi*k*t/T],{t,-T/2,0}]+
2/T*Integrate[(T-4*t)*Exp[-I*2*Pi*k*t/T],{t,0,T/2}]

                        I k Pi                                    I k Pi
          (-2 - I k Pi + E       (2 - I k Pi)) T        (2 + I k Pi + E       (-2 + I k Pi)) T
Out[3]= -------------------------------------     +    -----------------------------------------
                 I k Pi  2   2                                          2   2
                E          k  Pi                                       k  Pi

In[4]:= Simplify[%,Element[k,Integers]]

                   k             k               k
        (-1 + (-1) ) (2 - 2 (-1)  + I (1 + (-1) ) k Pi) T
Out[4]= ------------------------------------------------
                          k  2   2
                      (-1)  k  Pi
```

$$Xk(k) = \frac{\left[-1 + (-1)^k\right]\cdot\left[2 - 2\cdot(-1)^k + 1j\cdot\left[1 + (-1)^k\right]\cdot(k\cdot\pi)\right]\cdot T0}{(-1)^k\cdot k^2\cdot\pi^2}$$

$$Xk(k) = \frac{-4\cdot T0}{\left(k^2\cdot\pi^2\right)}\cdot\left[(-1)^k - 1\right]$$

$$(-1)^k - 1 = \begin{cases} -2 & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \begin{cases} \dfrac{8T}{k^2\pi^2} & k \quad odd \\ 0 & k \quad even \end{cases}$$

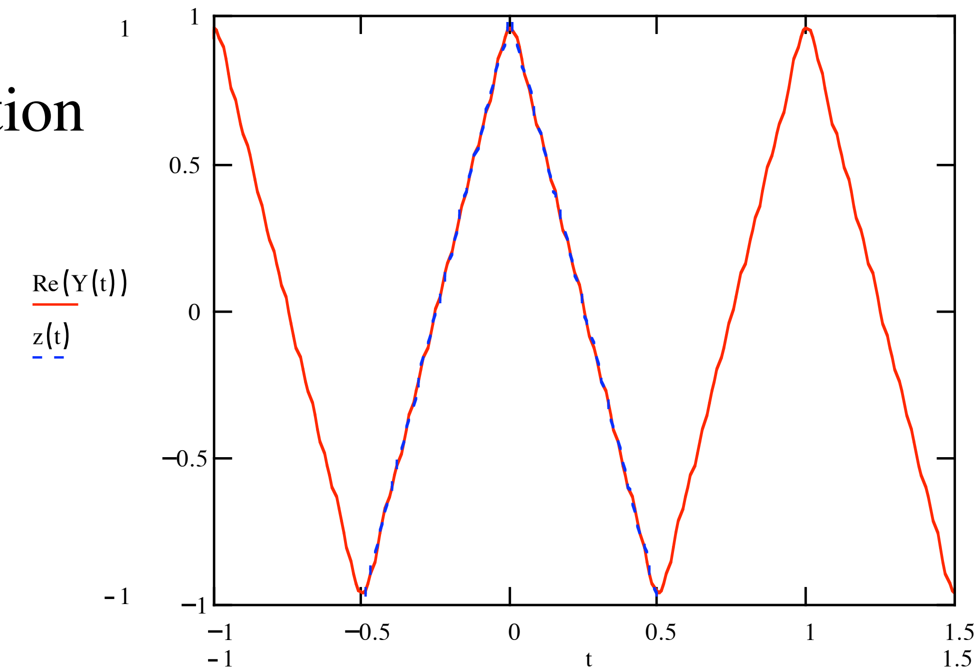
Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

Even functions $f(-x) = f(x)$ consist only of sums of cosines ($\phi = 0$)

Ex.

$$x(t) = \begin{cases} 4t + T_0 & -T_0/2 \le t < 0 \\ -4t + T_0 & 0 \le t < T_0/2 \end{cases}$$ even function



$$X_0 = 0 \qquad X_k = \begin{cases} \dfrac{8T}{k^2\pi^2} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \begin{cases} \dfrac{8T}{k^2\pi^2} e^{j0} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = \tfrac{8T}{\pi^2}\cos(2\pi f_0 t) + \tfrac{8T}{3^2\pi^2}\cos(2\pi 3 f_0 t) + \ldots$$

$f_0 = 1$

Properties of Fourier Series

Odd functions $f(-x) = -f(x)$ consist only of sums of sines ($\phi = -\frac{\pi}{2}$)

Even functions $f(-x) = f(x)$ consist only of sums of cosines ($\phi = 0$)

Ex.

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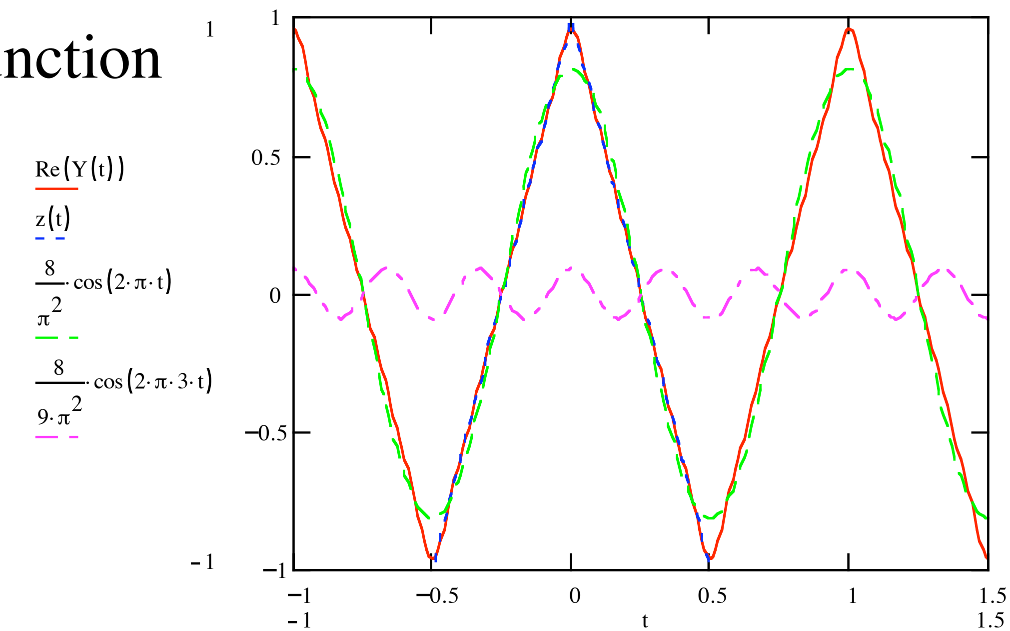


$$X_0 = 0 \qquad X_k = \begin{cases} \dfrac{8T}{k^2\pi^2} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \begin{cases} \dfrac{8T}{k^2\pi^2} e^{j0} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = \tfrac{8T}{\pi^2}\cos\left(2\pi f_0 t\right) + \tfrac{8T}{3^2\pi^2}\cos\left(2\pi 3 f_0 t\right) + \ldots$$

$f_0 = 1$

Properties of Fourier Series

Symmetric functions with $f(-x) = -f(x + T/2)$ only have odd harmonics

Symmetric functions with $f(-x) = f(x + T/2)$ only have even harmonics

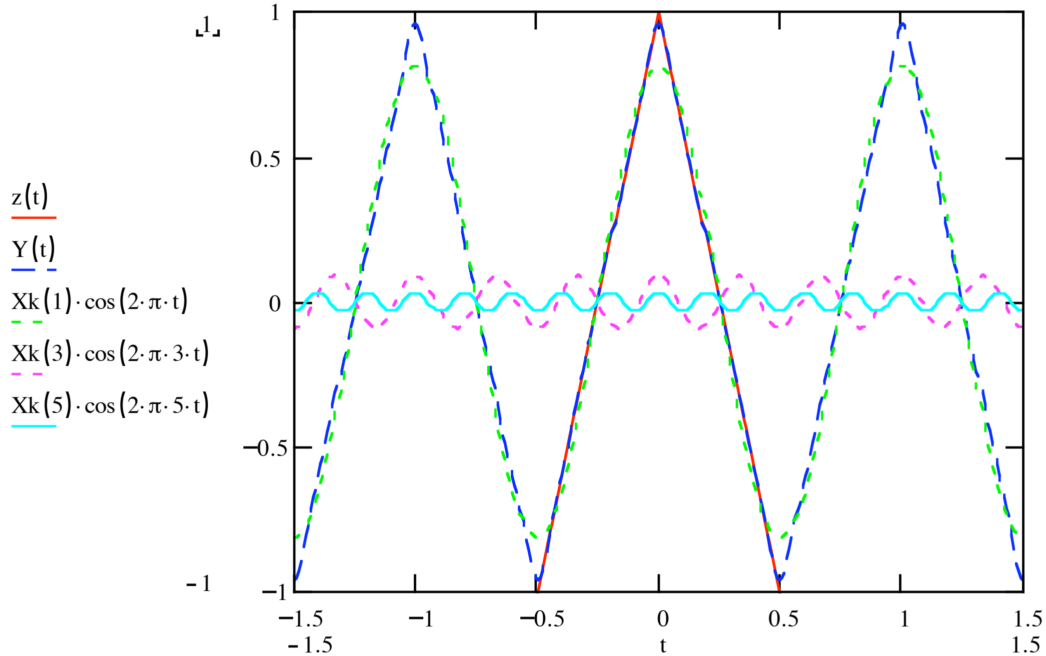
Ex. Triangle wave (only odd harmonics)

$$x(t) = \begin{cases} 4t + T_0 & -T_0/2 \le t < 0 \\ -4t + T_0 & 0 \le t < T_0/2 \end{cases}$$

$$X_0 = 0 \qquad X_k = \begin{cases} \frac{8T}{k^2\pi^2} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$X_k = \begin{cases} \frac{8T}{k^2\pi^2} e^{j0} & k \quad odd \\ 0 & k \quad even \end{cases}$$

$$x(t) = \tfrac{8T}{\pi^2}\cos(2\pi f_0 t) + \tfrac{8T}{3^2\pi^2}\cos(2\pi 3 f_0 t) + \ldots$$



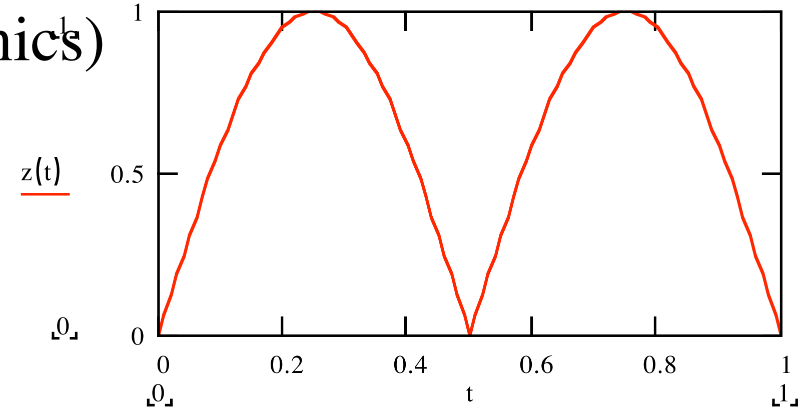
$f_0 = 1$

Properties of Fourier Series

Symmetric functions with $f(-x) = -f(x+T/2)$ only have odd harmonics

Symmetric functions with $f(-x) = f(x+T/2)$ only have even harmonics

Ex. Abs sine wave (only even harmonics)



$$x(t) = \left|\sin\left(\frac{2\pi}{T_0}t\right)\right| \quad 0 \le t < T_0$$

$f_0 = 1$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0}\left|\sin\left(\frac{2\pi}{T_0}t\right)\right|dt = \frac{1}{T_0}\int_0^{T_0/2}\sin\left(\frac{2\pi}{T_0}t\right)dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}-\sin\left(\frac{2\pi}{T_0}t\right)dt$$

```
In[2]:= 1/T*Integrate[Sin[2*Pi*t/T],{t,0,T/2}]+
1/T*Integrate[- Sin[2*Pi*t/T],{t,T/2,T}]

         2
Out[2]= --
        Pi
```

$$X_0 = \frac{2}{\pi}$$

$$X_k = \frac{2}{T_0}\int_0^{T_0}\left|\sin\left(\frac{2\pi}{T_0}t\right)\right| e^{-j2\pi kt/T_0}\,dt = \frac{2}{T_0}\int_0^{T_0/2}\sin\left(\frac{2\pi}{T_0}t\right) e^{-j2\pi kt/T_0}\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}\left(-\sin\left(\frac{2\pi}{T_0}t\right)\right) e^{-j2\pi kt/T_0}\,dt$$

```
In[5]:= 2/T*Integrate[Sin[2*Pi*t/T]*Exp[-I*2*Pi*k*t/T],{t,0, T/2}]+
2/T*Integrate[- Sin[2*Pi*t/T]*Exp[-I*2*Pi*k*t/T],{t,T/2,T}]
```

```
                  T             T         T
        2 (T + -------)    2 (------- + -----------)
                I k Pi          I k Pi   (2 I) k Pi
              E                E         E
Out[5]= ----------------- + -------------------------
                      2                        2
         (2 Pi - 2 k  Pi) T     (2 Pi - 2 k  Pi) T
```

```
In[6]:=  Simplify[%,Element[k,Integers]]
```

```
                   k 2
           (1 + (-1) )
Out[6]= -(------------)
                   2
            (-1 + k ) Pi
```

$$X_k = -\frac{\left(1+(-1)^k\right)^2}{\left(-1+k^2\right)\pi}$$

$$X_k = \begin{cases} 0 & k \quad odd \\ \dfrac{4}{\left(-1+k^2\right)\pi} & k \quad even \end{cases}$$

Properties of Fourier Series

Symmetric functions with $f(-x) = -f(x + T/2)$ only have odd harmonics

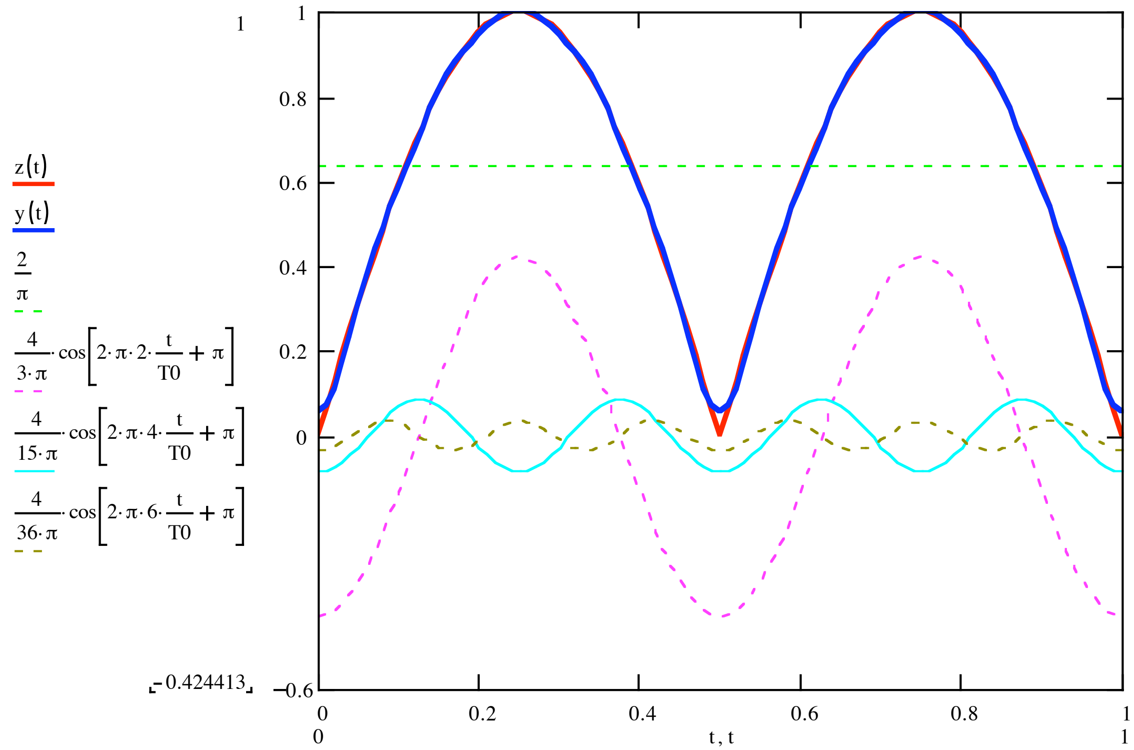
Symmetric functions with $f(-x) = f(x + T/2)$ only have even harmonics

Ex. Abs sine wave (only even harmonics)

$$x(t) = \left|\sin\left(\frac{2\pi}{T_0}t\right)\right| \quad 0 \le t < T_0$$

$$X_0 = \frac{2}{\pi}$$

$$X_k = \begin{cases} 0 & k \quad odd \\ \dfrac{4}{\left(-1+k^2\right)\pi} & k \quad even \end{cases}$$



$$x(t) = \frac{2}{\pi} + \tfrac{4}{3\pi}\cos\left(2\pi 2 f_0 t + \pi\right) + \tfrac{4}{15\pi}\cos\left(2\pi 4 f_0 t + \pi\right) + \ldots$$

How it works

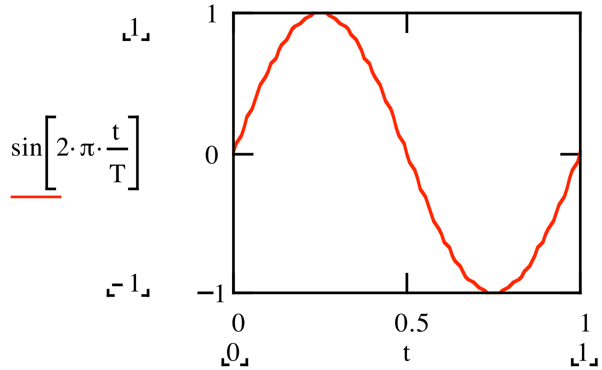
Ex. sine wave

$$x(t) = \sin\left(\frac{2\pi}{T_0} t\right) \quad 0 \le t < T_0$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} \sin\left(\frac{2\pi}{T_0} t\right)dt$$

$$X_0 = \frac{-1}{2\pi}\cos\left(\frac{2\pi}{T_0} t\right)\Bigg|_0^{T_0} = 0$$



Average of a sinusoid over one period is equal to zero.

Ex. sine wave

$$x(t) = \sin\left(\frac{2\pi}{T_0} t\right) \quad 0 \le t < T_0$$

$$Xk = \frac{2}{T} \cdot \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\, dt$$

$$Xk = \frac{2}{T} \cdot \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\left(\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right) + 1j\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\right) dt$$

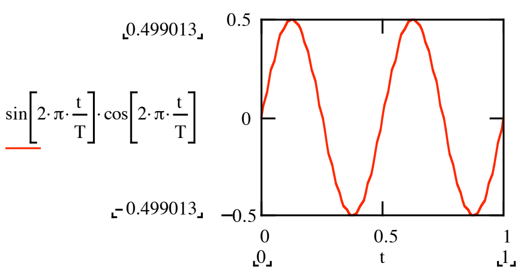
$$Xk = \frac{2}{T} \cdot \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right) dt + \frac{2\cdot 1j}{T}\cdot \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right) dt$$

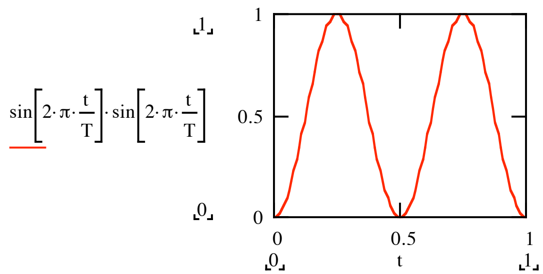
Ex. sine wave

$$Xk = \frac{2}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\,dt + \frac{2\cdot 1j}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\,dt$$

$k = 1$

$$X_1 = \frac{2}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(2\cdot\pi\cdot\frac{t}{T}\right)\,dt - \frac{2\cdot 1j}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\,dt$$





$$\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\,dt = 0 \quad\blacksquare \qquad \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\,dt = 0.5 \quad\blacksquare$$

$$X_1 = 0 - \frac{2\cdot 1j}{T}\cdot\frac{1}{2}\cdot T$$

$$X_1 = -1i$$

$$X_1 = 1\cdot e^{\frac{-\pi}{2}}$$

$$\cos(\omega t + \phi) = \cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t)$$

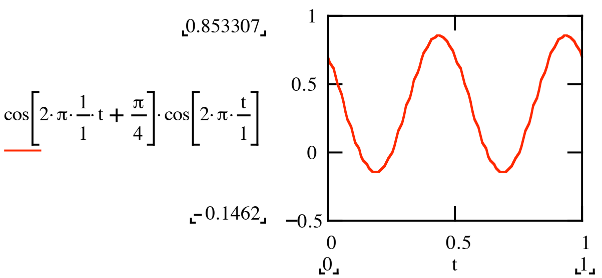
Ex. cos wave + phase

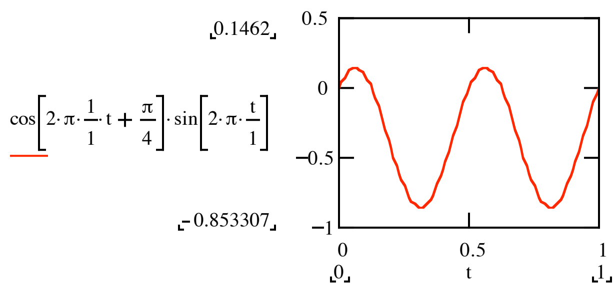
$$\cos(\omega t+\phi)=\cos(\phi)\cos(\omega t)-\sin(\phi)\sin(\omega t)$$

$$X_k = \frac{2}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{1}{T}+\phi\right)\cdot\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)dt + \frac{2\cdot j}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{1}{T}+\phi\right)\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)dt$$

$$k=1$$

$$X_k = \frac{2}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}+\phi\right)\cdot\cos\left(2\cdot\pi\cdot\frac{t}{T}\right)dt + \frac{-2\cdot j}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}+\frac{\pi}{4}\right)\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)dt$$





$$\frac{2}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}+\phi\right)\cdot\cos\left(2\cdot\pi\cdot\frac{t}{T}\right)dt = \cos(\phi)$$

$$\frac{2\cdot j}{T}\cdot\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}+\phi\right)\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)dt = -1j\cdot\sin(\phi)$$

$$X_1 = \cos(\phi) - 1j\cdot\sin(\phi)$$

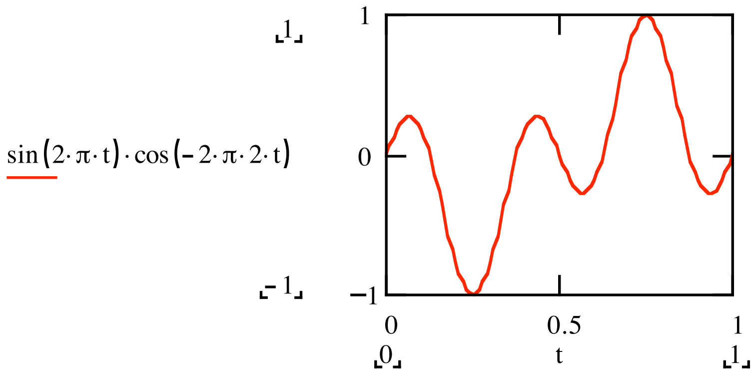
$$X_1 = 1\cdot e^{-j\cdot\phi}$$

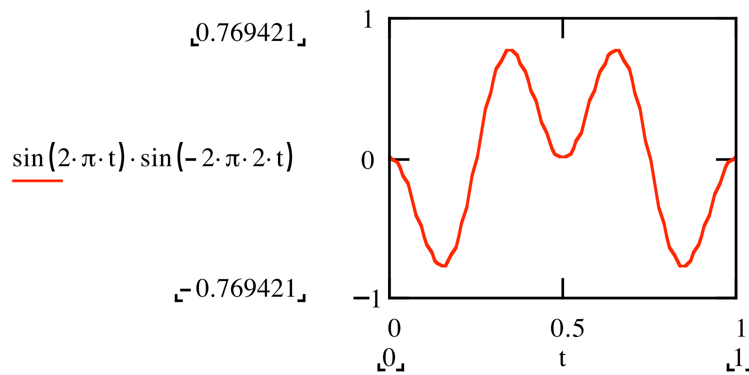
Ex. sine wave

$$Xk = \frac{2}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)dt + \frac{2\cdot 1j}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)dt$$

$k = 2$

$$X_2 = \frac{2}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(-2\cdot\pi\cdot 2\cdot\frac{t}{T}\right)dt + \frac{2\cdot 1j}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(-2\cdot\pi\cdot 2\cdot\frac{t}{T}\right)dt$$





$$\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(2\cdot\pi\cdot 2\cdot\frac{t}{T}\right)dt = 0$$

$$\int_0^T \cos\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(2\cdot\pi\cdot 2\cdot\frac{t}{T}\right)dt = 0$$

$X_2 = 0$

Ex. sine wave

$k \neq 1$

$$Xk = \frac{2}{T} \cdot \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\, dt + \frac{2\cdot 1j}{T}\cdot\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(-2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\, dt$$

$$\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\sin\left(2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\, dt = 0 \qquad \int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot\cos\left(2\cdot\pi\cdot k\cdot\frac{t}{T}\right)\, dt = 0 \quad \blacksquare$$

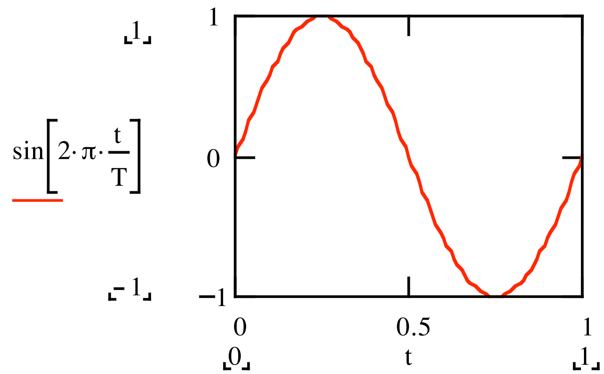
$$X_k = 0$$

```
In[11]:=
2/T*Integrate[Sin[2*Pi*t/T]*Sin[2*Pi*k*t/T],{t,0,T}]+
2*I/T*Integrate[Sin[2*Pi*t/T]*Cos[2*Pi*k*t/T],{t,0,T}]

                            2
          (2 I) Sin[k Pi]    Sin[2 k Pi]
Out[11]= ---------------- + ------------
                 2                  2
         Pi - k  Pi         (-1 + k ) Pi

In[12]:=  Simplify[%,Element[k,Integers]]

Out[12]= 0
```

How it works

Ex. sine wave

$$x(t) = \sin\left(\frac{2\pi}{T_0} t\right) \quad 0 \le t < T_0$$



$$X_0 = 0$$

$$X_k = \begin{cases} 1e^{-\frac{\pi}{2}} & k = 1 \\ 0 & k > 1 \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right)$$

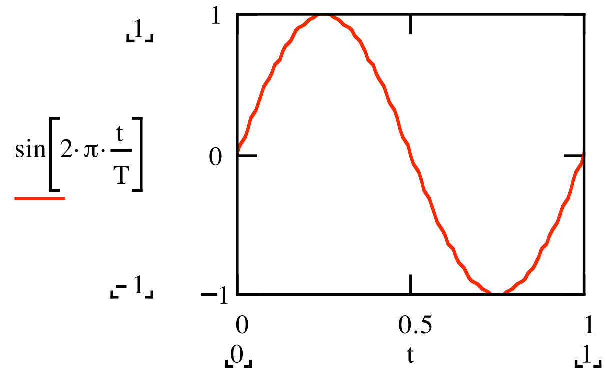
$$x(t) = \sin(2\pi f_0 t)$$

$$x(t) = \sin\left(\frac{2\pi}{T_0} t\right) \quad 0 \le t < T_0$$

How it works

Ex. sine wave

$$x(t) = \sin\left(\frac{2\pi}{T_0} t\right) \quad 0 \le t < T_0$$



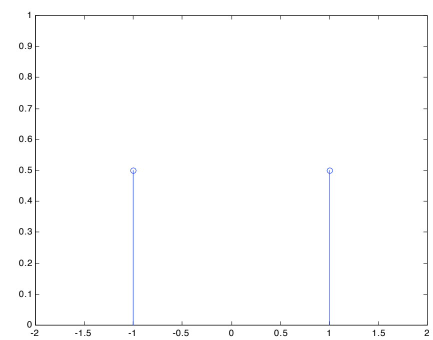
$$X_0 = 0$$

$$X_k = \begin{cases} 1e^{-\frac{\pi}{2}} & k = 1 \\ 0 & k = 2, 3, \ldots \end{cases}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = X_0 + \mathrm{Re}\left\{\sum_{k=1}^{\infty} X_k e^{j2\pi k f_0 t}\right\}$$

$$x(t) = \cos\left(2\pi f_0 t - \tfrac{\pi}{2}\right)$$

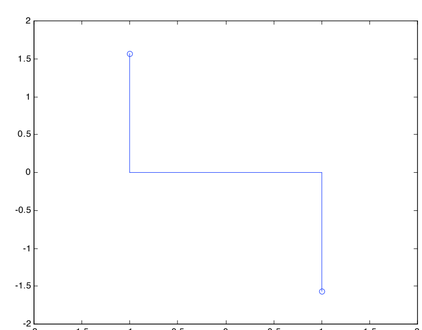
$$x(t) = \sin(2\pi f_0 t)$$



$$x(t) = Z_0 + \sum_{k=1}^{\infty} Z_k \left(\frac{e^{j2\pi k f_0 t} + e^{-j2\pi k f_0 t}}{2}\right)$$

$$Z_0 = 0$$

$$Z_k = \frac{X_k}{2} = \begin{cases} \frac{1}{2} e^{-\frac{\pi}{2}k} & k = \pm 1 \\ 0 & k = \pm 2, \pm 3 \ldots \end{cases}$$



Harmonic sinusoid

$$X_k = \frac{2}{T_0}\int_0^{T_0} A\cos\left(\frac{2\pi}{T_0}mt\right)e^{-j2\pi kt/T_0}\,dt$$

$$X_k = \begin{cases} A & if \quad m = k \\ 0 & if \quad m \neq k \end{cases}$$

Sum of harmonic sinusoids

$$X_k = \frac{2}{T_0}\int_0^{T_0}\left[\sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t)\right] e^{-j2\pi kt/T_0}\,dt$$

$$X_k = \frac{2}{T_0}\int_0^{T_0}\left[A_1\cos(2\pi f_0 t) + A_2\cos(2\pi 2 f_0 t) + A_3\cos(2\pi 3 f_0 t) + \ldots\right] e^{-j2\pi kt/T_0}\,dt$$

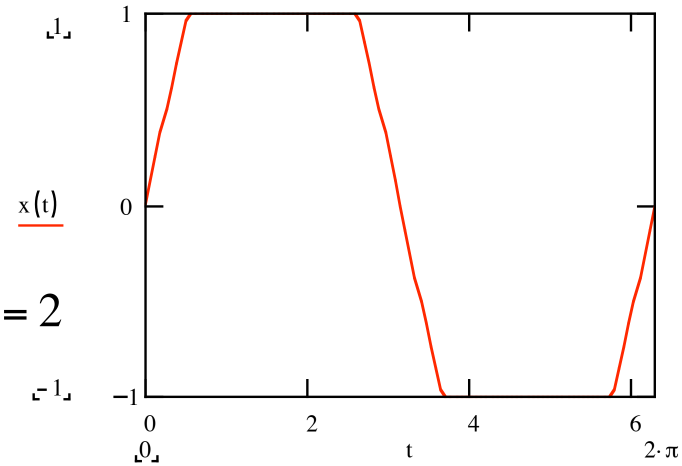
$$X_k = \underbrace{\frac{2}{T_0}\int_0^{T_0} A_1\cos(2\pi f_0 t) e^{-j2\pi kt/T_0}\,dt}_{\begin{matrix} A_1 & if & k=1 \\ 0 & if & k\neq 1\end{matrix}} + \underbrace{\frac{2}{T_0}\int_0^{T_0} A_2\cos(2\pi 2 f_0 t) e^{-j2\pi kt/T_0}\,dt}_{\begin{matrix} A_2 & if & k=2 \\ 0 & if & k\neq 2\end{matrix}} + \underbrace{\frac{2}{T_0}\int_0^{T_0} A_3\cos(2\pi 3 f_0 t) e^{-j2\pi kt/T_0}\,dt}_{\begin{matrix} A_3 & if & k=3 \\ 0 & if & k\neq 3\end{matrix}} + .$$

$$X_k = A_k$$

Clipped Sinewave

$A = 2$

$T_0 = \frac{1}{f_0} = 2$

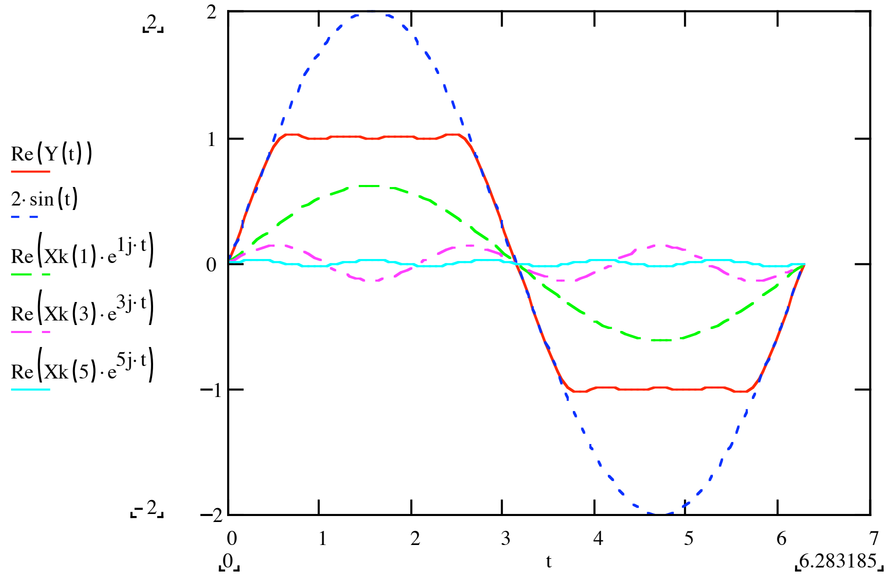


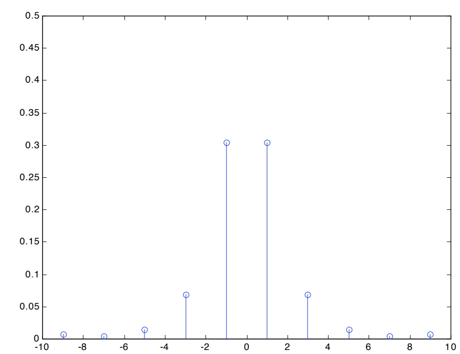
$$X_k = \begin{cases} 1 & \text{if} \quad A\sin(2\pi f_0 t) > 1 \\ -1 & \text{if} \quad A\sin(2\pi f_0 t) < 1 \\ A\sin(2\pi f_0 t) & \text{otherwise} \end{cases}$$

Xo = 0

$$Xk = \frac{2}{T}\cdot\int_{0}^{T1} A\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\,dt + \frac{2}{T}\cdot\int_{T1}^{T2} e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\,dt + \frac{2}{T}\cdot\int_{T2}^{T3} A\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\,dt + \frac{2}{T}\cdot\int_{T3}^{T4} -1\cdot e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\,dt + \frac{2}{T}\cdot\int_{T4}^{T} A\cdot\sin\left(2\cdot\pi\cdot\frac{t}{T}\right)\cdot e^{1j\cdot -2\cdot\pi\cdot k\cdot\frac{1}{T}\cdot t}\,dt$$

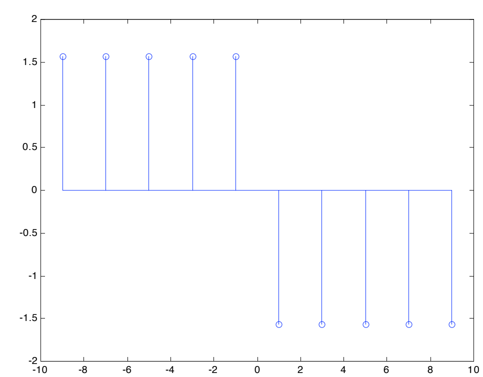
Clipped Sinewave





amp

Clipping adds harmonics
which distorts the pure tone.
“richer” sound



phase

```
t=0:1/8192:0.5;
y=sin(2*pi*440*t);
sound(y/2)
sound(y)
sound(2*y)
sound(5*y)
sound(10*y)
```