22.02 – Introduction to Applied Nuclear Physics

Problem set # 1

Issued on Thursday Feb. 16, 2012. Due on Wednesday Feb. 22, 2012

Useful quantities are: $\frac{e^2}{\hbar c} = \frac{1}{137}$, $\hbar c = 197$ MeV fm.

Problem 1: Nuclear physics nomenclature

Assume we have 147 neutrons and 95 protons in a box, and want to form some nuclides.

a) If we combine all the nucleons, what are N, Z and A of the nuclide we obtain? What is its symbol?

b) Now we use some of the nucleons to form an alpha particle and the rest to form a second nuclide **X**. What is this nuclide? (state its N, Z and A numbers as well as the symbol).

c) We now form one nucleus with 4 protons and 4 neutrons and use the rest of the nucleons to form a second nuclide. What are the two nuclides? (state their N, Z and A numbers as well as the symbols).

Problem 2: Nuclear Binding Energy

a) When we put together the nucleons to form the nuclides we considered in Problem 1: some energy is usually gained. This energy is the *binding* energy of each nuclide. Compute the total binding energy and binding energy per nucleon using both the measured data and the semi-empirical mass formula for the 5 nuclides considered in Problem 1:.

b) What is the most favorable combination of nuclides, a) b) or c)?

Problem 3: Coulomb potential energy

Consider the two possible combinations of nuclide in Problem 1: b) and c). To compare their energies to the bound state in Problem 1: a) we should consider not only the binding energy but also the Coulomb interaction between the two nuclides in each combination. In case b) for example, the alpha particle interacts with the nuclide **X** via the Coulomb interaction. To simplify the calculation, we assume both nuclides to be point particles at a distance $d = R_\alpha + R_X$ of each other, where $R_i \approx 1.25 \times A_i^{1/3}$ is the nucleus radius.

a) What is the Coulomb potential energy (in MeV) between each pair of nuclides considered in Problem 1: b) and c)? [Note: we will study in the following that this Coulomb repulsion is very important in determining e.g., if a nuclide will decay emitting an alpha particle]

Problem 4: Operators

We have seen in the lectures some examples of quantum mechanical operators. For example in 1D we have the

1. Position operator $\hat{x}$, $\hat{x}[\varphi(x)] = x\varphi(x)$.
2. Momentum operator $\hat{p}_x = -i\hbar\frac{d}{dx}$
3. Energy operator for a free particle: $\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

Other examples of operators are the following:

4. I, Identity operator: $I[\varphi(x)] = \varphi(x)$ (it leaves the function unchanged)
5. F, multiplication by the function $F(x)$: $F[\varphi(x)] = F(x)\varphi(x)$
6. A, annihilation operator: $A[\varphi(x)] = 0$
7. B, division by the number 21: $B[\varphi(x)] = \frac{1}{21}\varphi(x)$
8. C, operator that changes any function $\varphi(x)$ to the number 33: $C[\varphi(x)] = 33$

a) For all of the eight operators $\mathcal{O}$ above, f nd:
(i) The square of the operators, $\mathcal{O}^2$ [e.g. $\hat{x}^2[\varphi(x)] = x^2\varphi(x)$]
(ii) The inverse of the operators (when existing). The inverse of an operator $\mathcal{O}^{-1}$ is such that for any function φ the operator is applied to we have

$$\mathcal{O}^{-1}\mathcal{O}\varphi(x) = \mathcal{O}\mathcal{O}^{-1}\varphi(x) = I\varphi(x) = \varphi(x).$$

For example, $\hat{x}^{-1}[\varphi(x)] = \frac{1}{x}\varphi(x)$ which exists only for $x \neq 0$.

b) An operator is linear if $\mathcal{O}[a\varphi_1(x) + b\varphi_2(x)] = a\mathcal{O}[\varphi_1] + b\mathcal{O}[\varphi_2]$. Which of the 8 operators above is linear?

Problem 5: Wavelengths and quantum effects

(Solved) Derive the following expression relating the de Broglie wavelength of a particle of mass m with its (non-relativistic) energy and velocity, v, from the fundamental relation, $p = \hbar k = mv$:

$$\lambda = 2\pi \frac{\hbar c}{mc^2}\sqrt{\frac{mc^2}{2E}}$$

Solution:

The particle momentum p is related it to the wavelength, via its relation to the wavenumber k:

$$p = \hbar k = \hbar \frac{2\pi}{\lambda},$$

and it is related to the kinetic energy E by:

$$p = \sqrt{2mE}$$

Equating these two expressions, we can write

$$\lambda = \frac{2\pi\hbar}{\sqrt{2mE}} = 2\pi\frac{\hbar c}{mc^2}\sqrt{\frac{mc^2}{2E}}$$

Note that this expression makes it easier to evaluate λ through energy equivalents and universal constant combinations.

a) A nuclear reactor produces **fast** neutrons (with energy $\sim$ 1MeV) which are then slowed down to **thermal** neutrons (with energy of order $E \sim 0.025$eV, comparable to their thermal energy at room temperature). In research reactors, both types of neutrons could be selected to exit through a port and used in scattering experiments to study crystals. Crystal lattice spacing is usually a few angstrom and to get information about the crystal in a scattering experiment the radiation wavelength should be on the same order of the lattice spacing. Would you select fast or thermal neutrons for a scattering experiment?

[Calculate the de Broglie wavelength in both cases to give a more quantitative answer.]

b) One of the largest object for which diffraction effects have been observed are buckyballs [see Nature 401, 680-682 (1999)], a molecule containing 60 Carbon atoms. In the experiment, the buckyball velocity was $v \approx 220$m/s. What was the wavelength of the buckyball?

[Compare it to the diffraction grating used in the experiment consisting of 50-nm-wide slits with a 100-nm period.]

Problem 6: Eigenvalue equations

a) What is an eigenvalue equation? Write two examples of eigenvalue equations (no need to solve them).

b) A rotation in a plane (in a 2D geometric space) is given by an operator represented by the matrix:

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

Find the eigenvalues and eigenvectors of this operator, in the case $\vartheta = \pi/3$.

c) Consider a vector in a plane. The vector forms a 60° angle with the x-axis (in the xy plane).
i) Write an expression for the vector with respect to the basis given by the x and y axis.
ii) Now write the same vector with respect to axes rotated by 45° with respect to the x and y axes.

d) Consider a classical oscillator in 1D. The force acting on the oscillator when it's away from its equilibrium position by x is $F = -kx$. Write the equation of motion for the oscillator and explain why this can be considered as an eigenvalue equation. Solve for the eigenvalues and eigenfunctions of the oscillator.

Problem 7: Dirac Delta Function

Prove the following properties of the Dirac Delta Function:

(a – Solved) $\delta(x) = \delta(-x)$

Solution:

To prove a property of the delta function, you need to evaluate the integrals of the delta functions with an arbitrary function $f(x)$,

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(-x) f(x) dx &= -\int_{\infty}^{-\infty} \delta(x') f(-x') dx' \qquad (1) \\ &= \int_{-\infty}^{\infty} \delta(x') f(-x') dx' \\ &= f(0) \end{aligned}$$

In the last step we used the change of variables $x' = -x$ in (1). Since $\int \delta(x) f(x) dx = f(0)$ (by definition of the delta function), we have shown that $\delta(-x) = \delta(x)$.

b) $\delta(ax) = \frac{1}{a}\delta(x)$ (where a is a constant).

c) $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$

Problem 8: Fourier Transform

a) Another property of the Dirac Delta function is that $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$. Using this result, what is the Fourier transform of $\cos(kx)$?