22.101 Applied Nuclear Physics (Fall 2006)

Lecture 13 (10/30/06)

Radioactive-Series Decay

References:

R. D. Evans, *The Atomic Nucleus* (McGraw-Hill, New York, 1955).
W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967).

We begin with an experimental observation that in radioactive decay that the probability of a decay during a small time interval Δt, which we will denote as $P(\Delta t)$, is proportional to Δt. Given this as a fact one can write

$$P(\Delta t) = \lambda \Delta t \tag{12.1}$$

where λ is the proportionality constant which we will call the *decay constant.* Notice that this expression is meaningful only when $\lambda \Delta t < 1$, a condition which defines what we mean by a small time interval. In other words, $\Delta t < 1/\lambda$, which will turn out to be the mean life time of the radioisotope.

Suppose we are interested in the survival probability S(t), the probability that the radioisotope does not decay during an arbitrary time interval t. To calculate S(t) using (12.1) we can take the time interval t and divide it into many small, equal segments, each one of magnitude Δt. For a given t the number of such segments will be $t / \Delta t = n$. To survive the entire time interval t, we need to survive the first segment $(\Delta t)_1$, then the next segment $(\Delta t)_2$, …, all the way up to the nth segment $(\Delta t)_n$. Thus we can write

$$S(t) = \prod_{i=1}^{n} \left[1 - P((\Delta t)_i)\right]$$

$$= \left[1 - \lambda(t/n)\right]^n \rightarrow e^{-\lambda t} \tag{12.2}$$

where the arrow indicated the limit of $n \to \infty$, $\Delta t \to 0$. Unlike (12.1), (12.2) is valid for any t. When λt is sufficiently small compared to unity, it reduces to (12.1) as expected. Stated another way, (12.2) is extension of 1 – P(t) for arbitrary t. One should also notice a close similarity between (12.2) and the probability that a particle will go a distance x without collision, $e^{-\Sigma x}$, where Σ is the macroscopic collision cross section (recall Lec1). The role of the decay constant λ in the probability of no decay in a time t is the same as the macroscopic cross section Σ in the probability of no collision in a distance x. The exponential attenuation in time or space is quite a general result, which one encounters frequently. There is another way to derive it. Suppose the radioisotope has not decayed up to a time interval of t_1, for it to survive the next small segment Δt the probability is just 1 - P(Δt) = 1 - $\lambda \Delta t$. Then we have

$$S(t_1 + \Delta t) = S(t_1)\left[1 - \lambda \Delta t\right] \tag{12.3}$$

which we can rearrange to read

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = -\lambda S(t) \tag{12.4}$$

Taking the limit of small Δt, we get

$$\frac{dS(t)}{dt} = -\lambda \tag{12.5}$$

which we can readily integrate to give (12.2), since the initial condition in this case is S(t=0) = 1.

The decay of a single radioisotope is described by S(t) which depends on a single physical constant λ. Instead of λ one can speak of two equivalent quantities, the half life $t_{1/2}$ and the mean life τ. They are defined as

$$S(t_{1/2}) = 1/2 \quad \to \quad t_{1/2} = \ell n 2 / \lambda = 0.693 / \lambda \tag{12.6}$$

and $$\tau = \frac{\int_0^\infty dt' t' S(t')}{\int_0^\infty dt' S(t')} = \frac{1}{\lambda} \qquad (12.7)$$

Fig. 12.1 shows the relationship between these quantities and S(t).

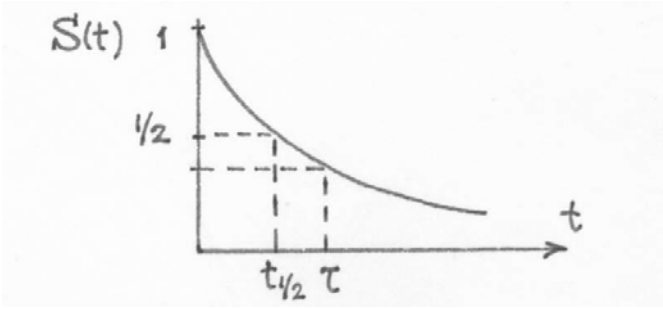


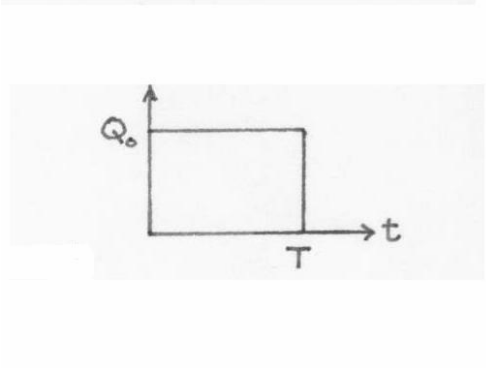
Fig. 12.1. The half life and mean life of a survival probability S(t).

Radioactivity is measured in terms of the rate of radioactive decay. The quantity λ N(t), where N is the number of radioisotope atoms at time t, is called *activity*. A standard unit of radioactivity has been the *curie*, 1 Ci = 3.7 x 10^{10} disintegrations/sec, which is roughly the activity of 1 gram of Ra^{226}. Now it is replaced by the *becquerel* (Bq), 1 Bq = 2.7 x 10^{-11} Ci. An old unit which is not often used is the *rutherford* (10^6 disintegrations/sec).

Radioisotope Production by Bombardment

There are two general ways of producing radioisotopes, activation by particle or radiation bombardment such as in a nuclear reactor or an accelerator, and the decay of a radioactive series. Both methods can be discussed in terms of a differential equation that governs the number of radioisotopes at time t, N(t). This is a first-order linear differential equation with constant coefficients, to which the solution can be readily obtained. Although there are different situations to which one can apply this equation, the analysis

is fundamentally quite straightforward. We will treat the activation problem first. Let Q(t), the rate of production of the radioisotope, have the form shown in the sketch below.



This means the production takes place at a constant Q_o for a time interval (0, T), after which production ceases. During production, t < T, the equation governing N(t) is

$$\frac{dN(t)}{dt} = Q_o - \lambda N(t) \tag{12.8}$$

Because we have an external source term, the equation is seen to be inhomogeneous. The solution to (12.8) with the initial condition that there is no radioisotope prior to production, N(t = 0) = 0, is

$$N(t) = \frac{Q_o}{\lambda}\left(1 - e^{-\lambda t}\right), \quad \text{t < T} \tag{12.9}$$

For t > T, the governing equation is (12.8) without the source term. The solution is

$$N(t) = \frac{Q_o}{\lambda}(1 - e^{-\lambda T})e^{-\lambda(t-T)} \tag{12.10}$$

A sketch of the solutions (12.9) and (12.10) is shown in Fig. 12.2. One sees a build up of N(t) during production which approaches the asymptotic value of Q_o / λ, and after production is stopped N(t) undergoes an exponential decay, so that if $\lambda T >> 1$,

$$N(t) \approx \frac{Q_o}{\lambda} e^{-\lambda(t-T)} \tag{12.11}$$

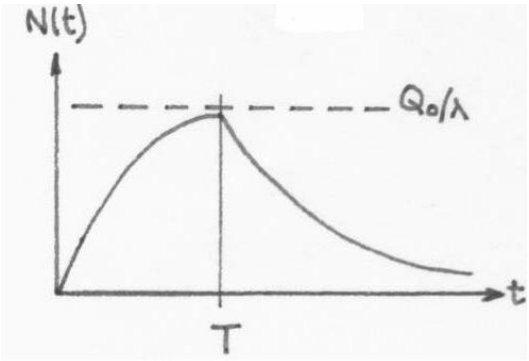


Fig. 12.2. Time variation of number of radioisotope atoms produced at a constant rate Qo for a time interval of T after which the system is left to decay.

Radioisotope Production in Series Decay

Radioisotopes also are produced as the product(s) of a series of sequential decays. Consider the case of a three-member chain,

$$N_1 \xrightarrow{\lambda_1} N_2 \xrightarrow{\lambda_2} N_3 \text{ (stable)}$$

where λ_1 and λ_2 are the decay constants of the parent (N_1) and the daughter (N_2) respectively. The governing equations are

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t) \tag{12.12}$$

$$\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t) \tag{12.13}$$

$$\frac{dN_3(t)}{dt} = \lambda_2 N_2(t) \tag{12.14}$$

For the initial conditions we assume there are N_{10} nuclides of species 1 and no nuclides of species 2 and 3. The solutions to (12.12) – (12.14) then become

$$N_1(t) = N_{10}e^{-\lambda_1 t} \tag{12.15}$$

$$N_2(t) = N_{10}\frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \tag{12.16}$$

$$N_3(t) = N_{10}\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1-e^{-\lambda_1 t}}{\lambda_1} - \frac{1-e^{-\lambda_2 t}}{\lambda_2}\right) \tag{12.17}$$

Eqs.(12.15) through (12.17) are known as the Bateman equations. One can use them to analyze situations when the decay constants λ_1 and λ_2 take on different relative values. We consider two such scenarios, the case where the parent is short-lived, $\lambda_1 >> \lambda_2$, and the opposite case where the parent is long- lived, $\lambda_2 >> \lambda_1$.

One should notice from (12.12) – (12.14) that the sum of these three differential equations is zero. This means that $N_1(t) + N_2(t) + N_3(t)$ = constant for any t. We also know from our initial conditions that this constant must be N_{10}. One can use this information to find $N_3(t)$ given $N_1(t)$ and $N_2(t)$, or use this as a check that the solutions given by (12.15) – (12.17) are indeed correct.

Series Decay with Short-Lived Parent

In this case one expects the parent to decay quickly and the daughter to build up quickly. The daughter then decays more slowly which means that the grand daughter will build up slowly, eventually approaching the initial number of the parent. Fig. 12.3 shows schematically the behavior of the three isotopes. The initial values of $N_2(t)$ and $N_3(t)$ can be readily deduced from an examination of (12.16) and (12.17).

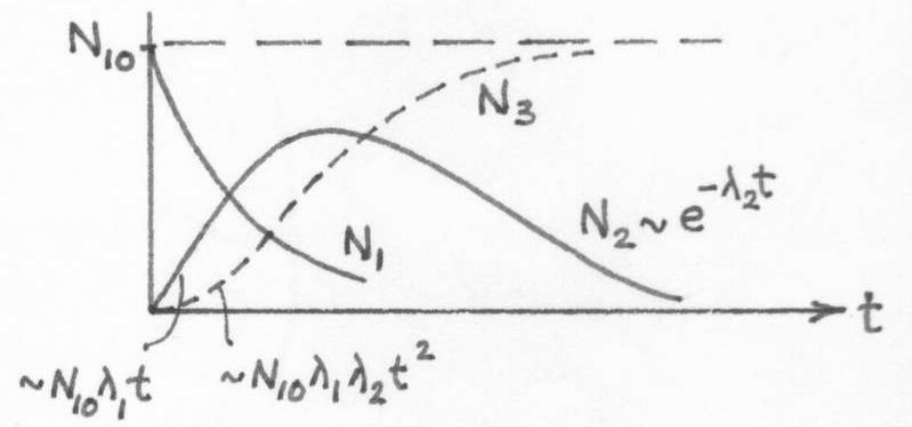


Fig. 12.3. Time variation of a three-member decay chain for the case $\lambda_1 >> \lambda_2$.

Series Decay with Long-Lived Parent

When $\lambda_1 << \lambda_2$, we expect the parent to decay slowly so the daughter and grand daughter will build up slowly. Since the daughter decays quickly the long-time behavior of the daughter follows that of the parent. Fig. 12.4 shows the general behavior

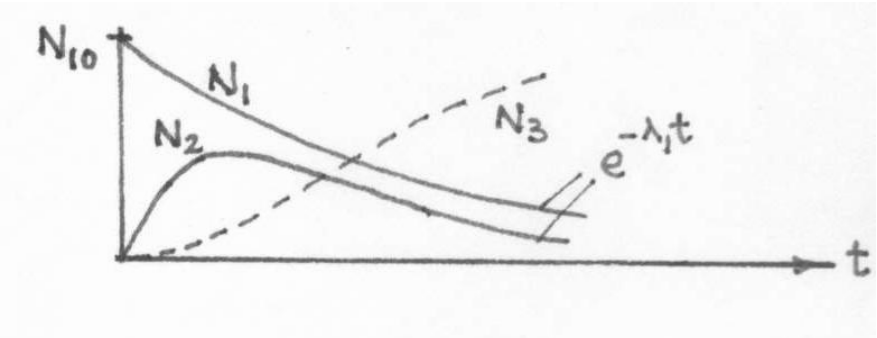


Fig. 12.4. Time variation of a three-member chain with a long-lived parent.

(admittedly the N_2 behavior is not sketched accurately). In this case we find

$$N_2(t) \approx N_{10} \frac{\lambda_1}{\lambda_2} e^{-\lambda t} \tag{12.18}$$

or $$\lambda_2 N_2(t) \approx \lambda_1 N_1(t) \qquad (12.19)$$

The condition of approximately equal activities is called *secular equilibrium.* Generalizing this to an arbitrary chain, we can say for the series

$$N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow ...$$

if $$\lambda_2 >> \lambda_1,\ \lambda_3 >> \lambda_1, \ldots$$

then $$\lambda_1 N_1 \approx \lambda_2 N_2 \approx \lambda_3 N_3 \approx ... \qquad (12.20)$$

This condition can be used to estimate the half life of a very long-lived radioisotope. An example is U^{238} whose half life is so long that it is difficult to determine by directly measuring its decay. However, it is known that $U^{238} \rightarrow Th^{234} \rightarrow \ldots \rightarrow Ra^{226} \rightarrow \ldots$, and in uranium mineral the ratio of $N(U^{238})/N(Ra^{226}) = 2.8 \times 10^6$ has been measured, with $t_{1/2}(Ra^{226}) = 1620$ yr. Using these data we can write

$$\frac{N(U^{238})}{t_{1/2}(U^{238})} = \frac{N(Ra^{226})}{t_{1/2}(Ra^{226})} \quad \text{or} \quad t_{1/2}(U^{238}) = 2.8 \times 10^6 \times 1620 = 4.5 \times 10^9 \text{ yr.}$$

In so doing we assume that all the intermediate decay constants are larger than that of U^{238}. It turns out that this is indeed true, and that the above estimate is a good result. For an extensive treatment of radioactive series decay, the student should consult Evans.