Final Exam

3 Hours. Closed Book. No electronic aids.

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1. Reduce the following ordinary differential equation to a first-order vector differential equations, which you should write out completely, in vector format.

$$\left(\frac{d^3y}{dx^3}\right)^3 - -\frac{d^2y}{dx^2} - y^2 = 0.$$

2. Consider an approximate discrete step in x and y, starting at $x = y = 0$ of the ODE $dy/dx = f(y, x)$. The Taylor expansion of the derivative function is

$$f(y, x) = f_0 + \frac{df_0}{dx}x + \frac{d^2f_0}{dx^2}\frac{x^2}{2!} + \dots \tag{1}$$

along the orbit. The approximate scheme is the following:

"Evaluate $y_1 = f_0\,\frac{x}{3}$, then $y_2 = f(y_1, \frac{x}{3})\,x$. The step is then $y = [kf_1 + (1-k)f_2]x$."
Find the value of k that makes this scheme accurate to second order as follows.

(a) Express the exact solution for $y(x)$ as a Taylor expansion.
(b) Express y_1 in terms of the Taylor expansion.
(c) Hence find an expression for y_2 and finally y complete to second order in x.
(d) Find the value of k that annihilates (makes zero) the second order term.

3. Consider the partial differential equation

$$\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial^2\psi}{\partial y^2} + 3\frac{\partial^2\psi}{\partial z^2} = 0.$$

(a) Is this equation parabolic, elliptic, or hyperbolic?
(b) Express the equation approximately in terms of discrete finite differences, centered on the grid point whose x, y, and z indices are (i, j, k), using only immediate adjacent point values (and i, j, k itself), on a mesh uniform in each coordinate direction, with point spacings Δx, $\Delta y = \Delta x$, $\Delta z = 3\Delta x$.
(c) If this difference equation is written as the sum over stencil points $\sum_n a_n\psi_n = 0$, what is the sum of the coefficients $\sum_n a_n$?

4. A diffusion equation in 2 dimensions with suitably normalized time units is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\partial\psi}{\partial t},$$

on a finite domain with fixed ψ on the boundary. It is to be advanced in time using an *implicit* scheme.

$$\boldsymbol{\psi}^{(n+1)} - \boldsymbol{\psi}^{(n)} = \Delta t\,\mathbf{D}\boldsymbol{\psi}^{(n+1)}.$$

where $\boldsymbol{\psi}^{(n)}$ denotes the value at the nth time step, and is a column vector of the values on a discrete grid. The matrix $\mathbf{D}$ represents the finite difference form of the spatial differential operator ∇^2. Let the dimensions of $\boldsymbol{\psi}$ be N.

(a) What is the minimum number of non-zero entries on each row of $\mathbf{D}$?

(b) Hence what is the minimum number of multiplications needed to evaluate $\mathbf{D}\boldsymbol{\psi}$.

(c) Show formally how the implicit time-step advance can actually be implemented, requiring a matrix inversion.

(d) What is the number of multiplications needed to perform each time-step advance using this implementation? By what factor is this bigger than the answer to (b)?

5. Divergence of acceleration in phase space.

(a) Prove that particles of charge q moving in a magnetic field $\mathbf{B}$ and hence subject to a force $q\mathbf{v} \times \mathbf{B}$, nevertheless have $\nabla_v.\mathbf{a} = 0$.

(b) Consider a frictional force that slows particles down in accordance with $\mathbf{a} = -K\mathbf{v}$, where K is a constant. What is the "velocity-divergence", of this acceleration, $\nabla_v.\mathbf{a}$? Does this cause the distribution function f to increase or decrease as a function of time?

(c) Write down (but don't attempt to solve!) the Boltzmann equation governing particles that have both magnetic force (a) and friction force (b).

6. Consider a one-group representation of neutron transport in a slab, one-dimensional, reactor of length $2L$. The reactor has uniform material properties; so that the steady diffusion equation becomes

$$-D_\phi \nabla^2 \Phi + (\Sigma_t - S)\Phi - \frac{1}{k} F\Phi = 0$$

where the diffusion coefficient D_ϕ, the total attenuation "macroscopic cross-section" Σ_t, the scattering and fission source terms S, F, are simply scalar constants. For convenience, write $\Sigma_t - S = \Sigma$. The eigenvalue k must be found for this equation.

The boundary conditions at $x = \pm L$ are that the flux satisfy $\Phi = 0$.

Formulate the finite-difference diffusion equation on a uniform mesh of N_x nodes; node spacing $\Delta x = 2L/(N_x - 1)$. Exhibit it in the form of a matrix equation

$$[\mathbf{M} - \frac{1}{k}\mathbf{F}]\boldsymbol{\Phi} = 0$$

And write out the matrix $\mathbf{M}$ explicitly for the case $N_x = 5$ (so $\mathbf{M}$ is 5×5), carefully considering the incorporation of the finite-difference boundary condition.

State in no more than a few sentences the significance of the eigenvalue k, and how one might find its value numerically.

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