22.38 - PS#7 Solutions

5.3) $f_H(h) = \frac{h}{\alpha^2} e^{-1/2 (h/\alpha)^2}$

given the wave height measurements, the liklihood function is:

$$L(h_1, \ldots h_n, \alpha) = \prod_i^n \frac{h}{\alpha^2} e^{-1/2 (h/\alpha)^2}$$

$$= \left(\prod_i h\right)\left(\frac{1}{\alpha^2}\right)^n e^{-1/2\alpha^2 \cdot \Sigma h_i^2}$$

to maximize L by optimizing α: $dL/d\alpha = 0$

$$\frac{dL}{d\alpha} = \Pi_i h_i \left[\left(-2n\alpha^{-(2n+1)} \cdot e^{-\frac{1}{2\alpha^2} \cdot \Sigma_i h_i^2}\right) + \left(\alpha^{-2n}\left(-\frac{\Sigma h_i^2}{2} e^{-\frac{1}{2\alpha^2} \Sigma_i h_i^2}\right)\right)\right] = 0$$

$\Sigma_i h_i^2 = 80.4$, $\Pi_i h_i = 16341$; $n = 20$

solving for α: $\alpha^2 = \frac{80.4}{20} = 4.02 \Rightarrow \boxed{\alpha = 2.005}$

5.5) a) Sample mean: $\bar{x} = \frac{1}{n}\sum_{i=1}^n V_i = \boxed{76.5}$

Sample variance: $s^2 = \frac{1}{n-1}\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right) = \boxed{14.27} \Rightarrow S = 3.78$

b) For 99% confidence interval $(.005 ; .995) = P(x)$

$\phi(z) = .995 \Rightarrow z = 2.58$; assume $s \sim \sigma$

interval: $-2.58 \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq 2.58 \Rightarrow \boxed{74.3 \leq \mu \leq 78.7}$

c) $\zeta_v^2 = \ln\left(1 + \frac{S_v^2}{\bar{x}_v^2}\right) = .0024$

$\lambda_v = \ln \bar{x} - 1/2\, \zeta_v^2$

$\Rightarrow$ $\boxed{\zeta = .049 \quad \lambda = 4.34}$

5.10) $\bar{X} = 20$ kips ; $\sigma = 3$ kips ; 90% confidence ; $n = 9$
assume normal dist.

a) $\phi(z) = .95 \Rightarrow z = 1.64$

$$-1.64 \le \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le 1.64 \quad \Rightarrow \quad \boxed{18.36 \le \mu \le 21.64}$$

b) $\phi(z) = .975 \Rightarrow z = 1.96$

$$\frac{\bar{X} - 18.36}{\sigma/\sqrt{9+n}} = 1.96 \quad \Rightarrow \text{ solving for } n = 3.85$$

∴ need 4 more samples for a 95% confidence

c) if the standard deviation is unknown, then we turn to the t-test, ~~[illegible]~~ because $n < 10$ (so we can't assume $s \simeq \sigma$):

$$s^2 = \frac{1}{n-1} \sum_{i}^{9} (x_i - \bar{x})^2 = \frac{1}{8}(84.5) = 10.56$$

$$P\left(-t_{\alpha/2, n-1} < \frac{\bar{X} - \mu}{s/\sqrt{n}} < t_{\alpha/2, n-1}\right) = 1 - \alpha \quad \Rightarrow \alpha = .1 \;;\; \alpha/2 = .05 \text{ for 90\% confidence}$$

$$t_{.05,8} = 1.86 \quad \Rightarrow \quad -1.86 < \frac{20 - \mu}{\sqrt{\frac{10.56}{9}}} < 1.86 \quad \Rightarrow \quad \mu_{90\%} = \boxed{(22.015 ; 17.985)}$$

5.14) assuming a normal distribution of the measurements,

a) $\bar{r}_1 = \sum_i r_i \cdot \frac{1}{n} = \underline{2.5}$; $s_1^2 = \frac{1}{n-1} \sum r_i^2 - n\bar{r}_1^2 = .01$

standard error: $\sigma_{r_1} \simeq s_1/\sqrt{n} = \underline{.045\text{cm}}$

r_2: $\bar{r}_2 = 1.5$; $s_2^2 = .01$; standard error $\sigma_{r_2} = \underline{.045\text{cm}}$

b) $\bar{A} = \pi(\bar{r}_1^2 - \bar{r}_2^2) = \pi(2.5^2 - 1.5^2) = \underline{12.566\text{ cm}^2}$

c) $A = \pi(r_1^2 - r_2^2) \Rightarrow \text{Var}A = \pi^2(\text{Var}(r_1^2) + \text{Var}(r_2^2))$

$\text{Var}(r^2) \simeq \text{Var}(r)\left(\frac{dg}{dr}\right)^2$

$\rightarrow \text{Var}(r_1^2) = \text{Var}(r_1)(2r_1)^2 = 4(2.5)^2 \cdot (.045)^2 = .051$

$\text{Var}(r_2^2) = \text{Var}(r_2)(2r_2)^2 = 4(1.5)^2 \cdot (.045)^2 = .0182$

$\Rightarrow \text{Var}A = \pi^2(.051 + .0182) = .679$

$\Rightarrow \sigma_A = \sqrt{\text{Var}A} = \underline{.82\text{ cm}^2}$

d) for $\bar{r}_1 - \mu_1 = \pm .07\text{cm}$ with 99% confidence

for $n < 10$ & unknown σ, use t-test

$\frac{\bar{r}_1 - \mu}{s/\sqrt{n}} = t_{.995, n-1} \Rightarrow \frac{.07}{.1} = \sqrt{n} \cdot t_{.995, n-1}$

iteration gives $n \simeq 17$ trials needed

$17 - 5 =$ <u>12 additional trials</u>

6.7) from part a, expect (assume) an exponential distribution with $a=0$ $\Rightarrow F(x) = 1-e^{-\lambda t}$

b) Minimum time to failure = 0 hrs

$MTTF = \frac{1}{n}\sum_i t_i = 626.2$ hrs

c)

Interval	Observed Freq	Expected Freq.	(n-e)^2/e
<100	9	5.903795762	1.623782575
100-300	9	9.322090973	0.011128683
300-700	8	11.6948799	1.16736021
700-1000	7	4.978573146	0.820750526
>1000	7	8.100660221	0.1495499
			3.772571893 = chi-squared value

Chi-sqared value for 1% significance and f=(5-2) 3 degrees of freedom: 11.3

$3.77 < 11.3$ ∴ an exponential distribution is a good fit for the given data

6.8)

Interval	Observed Freq	P(x)	Expected Freq.	(n-e)^2/e
0	6	0.301194212	6.023884238	9.46992E-05
1	8	0.361433054	7.228661086	0.082306213
>2	6	0.337372734	6.747454676	0.082799888
				0.165200801

where $P(x) = \frac{(\nu t)^x}{x!} e^{-\nu t}$; $\nu = 1.2$

χ^2-value for 1% significance and $f = (3-2) = 1$ is 31.82

$.165 \ll 31.82$ ∴ a Poisson distribution is a very good fit.