Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J — R. Parker
2 November 2006 — Due: 9 November 2006

Problem Set 6

NOTE: Due on the 9th, not 10th as on the version passed out in class. Also, see hints for Problems 1 and 2 on page 6.

Problem 1.

The analogy between Alfvén waves and waves on a stretched string is often made. In the latter case the relation between ω and k is $\omega = k\sqrt{\dfrac{T}{\rho}}$ where T is the tension in the string and ρ is the line density of the string (mass per unit length.)

a) Making the analogy between a circular tube of flux and the string, what would be the analogous ω-k relation for Alfvén waves?

b) The obvious answer from part a) differs from the Alfvén speed by a factor of $\sqrt{2}$. This difference is the result of ignoring the force on the flux tube due to the side pressure of the $\vec{B}$ field. Assume the flux tube has a perturbation proportional to $\sin(kz - \omega t)$. Calculate the restoring force density acting on the *side* of the tube and show, by adding it to the tensile force, the correct Alfvén speed is found.

Problem 2.

The ideal MHD equations together with the appropriate form of Maxwell's equations are:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{v} = 0 && \vec{E} + \vec{v}\times\vec{B} = 0 \\
&\rho\frac{d\vec{v}}{dt} = \vec{j}\times\vec{B} - \nabla p && \nabla\times\vec{B} = \mu_0\vec{j}, \quad \nabla\cdot\vec{B} = 0 \qquad (1) \\
&\frac{d(p\rho^{-\gamma})}{dt} && \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}
\end{aligned}
$$

As we have seen in class, the linearized version of these equations supports undamped Alfvén waves in a homogenous plasma immersed in a uniform magnetic field. For example, with $\vec{B} = \hat{z}B_0$ the $\vec{E}$-field in the shear wave can be written

$$\vec{E} = \hat{x}E_1\cos(k_A z - \omega t)$$

where $k_A = \omega / c_A$ and c_A is the Alfvén speed.

If the conductivity of the plasma is considered to be large but finite, the only modification to the above ideal equations is in the Ohm's law, which takes the form

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$$

As expected, the resistivity leads to damping of the Alfvén wave, as you are asked to show in this problem.

a) Assume the equilibrium plasma is homogenous, and without current or flow. Specifically assume

$$\begin{aligned} &\rho = \rho_0 + \rho_1(\vec{r},t) && \vec{v} = \vec{v}_1(\vec{r},t) \\ &\vec{B} = \vec{B}_0 + \vec{B}_1(\vec{r},t) && \vec{E} = \vec{E}_1(\vec{r},t) \\ &\vec{j} = \vec{j}_1(\vec{r},t) && p = p_0 + p_1(\vec{r},t) \end{aligned}$$

where the quantities labeled with a subscript 0 are constants while the quantities labeled with subscript 1 are small perturbations. Write down a complete linearized set of equations for the perturbed quantities.

b) Assume that we are interested in waves with space-time dependence given by $\exp(i\vec{k} \cdot \vec{r} - i\omega t)$. In the absence of resistivity, the perturbed magnetic field is given by

$$\vec{B}_1 = -\frac{\vec{k} \times (\vec{v}_1 \times \vec{B}_0)}{\omega}$$

In the presence of resistivity, the perturbed magnetic field can be written in the form

$$\vec{B}_1 = -\frac{\vec{k} \times (\vec{v}_1 \times \vec{B}_0)}{\omega^*}$$

Determine ω^*.

c) Again in the absence of resistivity, the velocity is determined by solution to the equation

$$\omega^2 \vec{V} = c_A^2 \left[\vec{k} \times (\vec{k} \times (\vec{V} \times \hat{z})) \right] \times \hat{z} + c_S^2 \vec{k}\vec{k} \cdot \vec{V} \,,$$

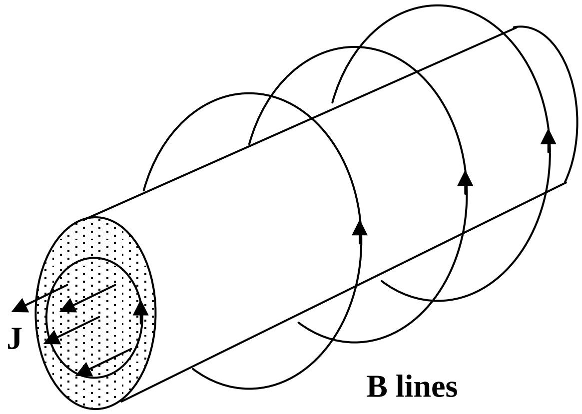
where we have taken the equilibrium magnetic field to be in the z-direction. In the presence of resisitivity, the same equation applies, but with $c_A \Rightarrow c_A^*$ and $c_S \Rightarrow c_S^*$. Determine c_A^* and c_S^*.

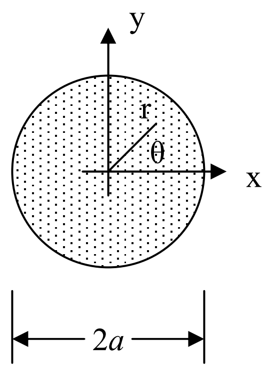
d) By making the substitution suggested in part c), determine the dispersion relation of the shear Alfvén mode propagating in the direction of $\vec{B}_0$ with $\vec{V} \perp \vec{B}_0$, including the effect of resistivity.

e) Assume that ω is real (corresponding to a source of this frequency.) Then, for the wave considered in part d), $\vec{k} = \hat{z}(k_r + ik_i)$, where k_r and k_i are the real and imaginary parts of the wavenumber. Determine k_r and k_i.

Problem 3.

In this problem, we'll analyze the m=0 stability of the straight z-pinch shown below. The plasma is assumed to obey the usual z-pinch equilibrium, $(p + B_{\theta 0}^2/2\mu_0)' + B_{\theta 0}^2/\mu_0 r = 0$ and there is no equilibrium flow. Modes having the form $\vec{v}_1 = \vec{V}(r)\exp(ikz + i\omega t)$ will be examined for their stability. For simplicity, we'll make the assumption that the plasma is incompressible, i.e., $\nabla \bullet \vec{v} = 0$, and assume the simple boundary condition $V_r(a) = 0$ where a is the column radius.





a) The perturbed magnetic field is given by

$$\vec{B}_1 = \frac{1}{i\omega}\nabla \times (\vec{v}_1 \times \vec{B}_0) = \frac{1}{i\omega}\left((\nabla \bullet \vec{B}_0)\vec{v}_1 + \vec{B}_0 \bullet \nabla \vec{v}_1 - (\nabla \bullet \vec{v}_1)\vec{B}_0 - \vec{v}_1 \bullet \nabla \vec{B}_0\right).$$

Using this relation and the incompressibility assumption, show that

$$\vec{B}_1 = \hat{\theta}\frac{1}{i\omega}\left(\frac{B_\theta}{r} - B_\theta'\right)V_r$$

b) The momentum equation can be written in the form

$$i\omega\rho_0\mu_0\vec{V} = \nabla \bullet (\vec{B}_0\vec{B}_1 + \vec{B}_1\vec{B}_0) - \nabla(\mu_0 p_1 + \vec{B}_0 \bullet \vec{B}_1)$$

By evaluating the terms on the right, show that the momentum equations become

$$i\omega\rho_0\mu_0 V_r = -2\frac{B_\theta B_{1\theta}}{r} - \left(B_\theta B_{1\theta} + \mu_0 p_1\right)'$$

$$i\omega\rho_0\mu_0 V_z = -ik\left(B_\theta B_{1\theta} + \mu_0 p_1\right)$$

c) Combine the results of parts a) and b) together with the constraint $\nabla \bullet \vec{V} = 0$ to get a single, second order differential equation for $\vec{V}(r)$. Put your result in the form

$$\left(\rho_0 \frac{1}{r}(rV_r)'\right)' + k^2 f(r,\omega^2)V_r = 0$$

d) By multiplying the result obtained in part c) by rV_r and integrating from $r = 0$ to $r = a$ (where V_r is assumed to vanish), show that the plasma will be unstable if

$$B_\theta' - \frac{B_\theta}{r} > 0$$

for $0 < r < a$.

Interpreting the above condition in terms of particle drifts leads to an apparent paradox since this requires the net particle drift (grad B plus curvature) to be in the $-z$ direction, which would be stabilizing according to the picture developed in class. The paradox can be resolved by calculating the energy involved in the perturbation. One can show that this is proportional to

$$\frac{2B_\theta^2}{r} + \mu_0 p'$$

It is interesting to note that *both* of these terms arise from field line curvature; thus arguments based only on "bad" curvature are incomplete and one must consider stabilizing as well as destabilizing effects of curvature to properly examine stability of MHD equilibria. (Nevertheless, a region of bad curvature is *necessary* for MHD instability.)

e) If $\frac{2B_\theta^2}{r} + p'\mu_0 < 0$ for $0 < r < a$, the energy argument shows that the plasma will be unstable. Show that this condition is equivalent to that found in part d).

Hints for Problem Set 6

Problem 1: Consider a flux tube with circular cross-section of radius *a*, where $a << 2\pi / k$. Let the displacement of the tube be given by $y = y_0 \sin(kz - \omega t)$, where $ky_0 << 1$. One can show (you don't need to) that the *restoring* force per unit length due to side pressure, i.e., pressure on the periphery, is

$$f_{\perp} = \frac{B^2}{2\mu_0} \pi a^2 \frac{1}{R}$$

where *R* is the radius of curvature of the field line. (We used this in the calculation of toroidal equilibrium (10/19 notes, page 6), but I skipped over the derivation.) Evaluate *R* and add to the restoring force due to the force per unit length due to the field line tension to get the equation of motion of the flux tube.

Problem 2: With the wave time-space dependence given by $\exp(i\vec{k} \bullet \vec{r} - i\omega t)$, a time derivative can be replaced by $-i\omega$ and the grad operator by $i\vec{k}$. So for example,

$$\nabla f = i\vec{k} f \ ,$$

$$\nabla \bullet \vec{A} = i\vec{k} \bullet \vec{A} \ ,$$

$$\nabla \times \vec{A} = i\vec{k} \times \vec{A} \ ,$$

etc. If this is the first time you have seen this, it would be worth a few minutes to convince yourself of these relationships.