Chapter 5

Electromagnetic Waves in Plasmas

5.1 General Treatment of Linear Waves in Anisotropic Medium

Start with general approach to waves in a linear Medium: Maxwell:

$$\nabla \wedge \mathbf{B} = \mu_o j + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad ; \quad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.1}$$

we keep all the medium's response *explicit* in $\mathbf{j}$. Plasma is (infinite and) uniform so we Fourier analyze in space and time. That is we seek a solution in which all variables go like

$$\exp i(\mathbf{k}.\mathbf{x} - \omega t) \quad \text{[real part of]} \tag{5.2}$$

It is really the linearised equations which we treat this way; if there is some equilibrium field OK but the equations above mean implicitly the perturbations $\mathbf{B}$, $\mathbf{E}$, $\mathbf{j}$, etc.

Fourier analyzed:

$$i\mathbf{k} \wedge \mathbf{B} = \mu_o \mathbf{j} + \frac{-i\omega}{c^2}\mathbf{E} \quad ; \quad i\mathbf{k} \wedge \mathbf{E} = i\omega \mathbf{B} \tag{5.3}$$

Eliminate $\mathbf{B}$ by taking $\mathbf{k}\wedge$ second eq. and $\omega\times$ 1st

$$i\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{E}) = \omega\mu_o \mathbf{j} - \frac{i\omega^2}{c^2}\mathbf{E} \tag{5.4}$$

So

$$\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{E}) + \frac{\omega^2}{c^2}\mathbf{E} + i\omega\mu_o \mathbf{j} = 0 \tag{5.5}$$

Now, in order to get further we must have some relationship between $\mathbf{j}$ and $\mathbf{E}(\mathbf{k}, \omega)$. This will have to come from solving the plasma equations but for now we can just write the most general *linear* relationship $\mathbf{j}$ and $\mathbf{E}$ as

$$\mathbf{j} = \boldsymbol{\sigma}.\mathbf{E} \tag{5.6}$$

$\boldsymbol{\sigma}$ is the 'conductivity tensor'. Think of this equation as a matrix e.g.:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{5.7}$$

This is a general form of Ohm's Law. Of course if the plasma (medium) is isotropic (same in all directions) all off-diagonal $\sigma' s$ are zero and one gets $\mathbf{j} = \sigma \mathbf{E}$.

Thus

$$\mathbf{k}(\mathbf{k}.\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} + i\omega\mu_o\boldsymbol{\sigma}.\mathbf{E} = 0 \tag{5.8}$$

Recall that in elementary E&M, dielectric media are discussed in terms of a dielectric constant ϵ and a "polarization" of the medium, $\mathbf{P}$, caused by modification of atoms. Then

$$\epsilon_o\mathbf{E} = \underbrace{\mathbf{D}}_{\text{Displacement}} - \underbrace{\mathbf{P}}_{\text{Polarization}} \quad \text{and} \quad \nabla.\mathbf{D} = \underbrace{\rho_{\text{ext}}}_{\text{externalcharge}} \tag{5.9}$$

and one writes

$$\mathbf{P} = \underbrace{\chi}_{\text{susceptibility}} \epsilon_o\mathbf{E} \tag{5.10}$$

Our case is completely analogous, except we have chosen to express the response of the medium in terms of current density, $\mathbf{j}$, rather than "polarization" $\mathbf{P}$ For such a dielectric medium, Ampere's law would be written:

$$\frac{1}{\mu_o}\nabla \wedge \mathbf{B} = \mathbf{j}_{\text{ext}} + \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\epsilon\epsilon_o\mathbf{E}, \quad \text{if } \mathbf{j}_{\text{ext}} = 0 \quad , \tag{5.11}$$

where the dielectric constant would be $\epsilon = 1 + \chi$.

Thus, the explicit polarization current can be expressed in the form of an equivalent dielectric expression if

$$\mathbf{j} + \epsilon_o\frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\sigma}.\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\epsilon_o\boldsymbol{\epsilon}.\mathbf{E} \tag{5.12}$$

or

$$\boldsymbol{\epsilon} = \mathbf{1} + \frac{\boldsymbol{\sigma}}{-i\omega\epsilon_o} \tag{5.13}$$

Notice the dielectric constant is a tensor because of anisotropy. The last two terms come from the RHS of Ampere's law:

$$\mathbf{j} + \frac{\partial}{\partial t}(\epsilon_o\mathbf{E}) \quad . \tag{5.14}$$

If we were thinking in terms of a dielectric medium with no explicit currents, only implicit (in ϵ) we would write this $\frac{\partial}{\partial t}(\epsilon\epsilon_o\mathbf{E})$; ϵ the dielectric constant. Our medium is possibly anisotropic so we need $\frac{\partial}{\partial t}(\epsilon_o\boldsymbol{\epsilon}.\mathbf{E})$ dielectric *tensor*. The obvious thing is therefore to define

$$\boldsymbol{\epsilon} = \mathbf{1} + \frac{1}{-i\omega\epsilon_o}\boldsymbol{\sigma} = \mathbf{1} + \frac{i\mu_o c^2}{\omega}\boldsymbol{\sigma} \tag{5.15}$$

Then

$$\mathbf{k}(\mathbf{k}.\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\boldsymbol{\epsilon}.\mathbf{E} = 0 \tag{5.16}$$

and we may regard $\boldsymbol{\epsilon}(\mathbf{k}, \omega)$ as the *dielectric tensor*.
Write the equation as a tensor multiplying $\mathbf{E}$:

$$\mathbf{D}.\mathbf{E} = 0 \tag{5.17}$$

with

$$\mathbf{D} = \{\mathbf{kk} - k^2\mathbf{1} + \frac{\omega^2}{c^2}\boldsymbol{\epsilon}\} \tag{5.18}$$

Again this is a matrix equation i.e. 3 simultaneous homogeneous eqs. for E.

$$\begin{pmatrix} D_{xx} & D_{xy} & ... \\ D_{yx} & ... & ... \\ ... & ... & D_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \tag{5.19}$$

In order to have a non-zero $\mathbf{E}$ solution we must have

$$\det \mid \mathbf{D} \mid = 0. \tag{5.20}$$

This will give us an equation relating $\mathbf{k}$ and ω, which tells us about the possible wavelengths and frequencies of waves in our plasma.

5.1.1 Simple Case. Isotropic Medium

$$\boldsymbol{\sigma} = \sigma \ \mathbf{1} \tag{5.21}$$

$$\boldsymbol{\epsilon} = \epsilon \ \mathbf{1} \tag{5.22}$$

Take $\mathbf{k}$ in z direction then write out the Dispersion tensor $\mathbf{D}$.

$$\mathbf{D} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & kk \end{pmatrix}}_{\mathbf{kk}} - \underbrace{\begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^2 \end{pmatrix}}_{k^2\mathbf{1}} + \underbrace{\begin{pmatrix} \frac{\omega^2}{c^2}\epsilon & 0 & 0 \\ 0 & \frac{\omega^2}{c^2}\epsilon & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{pmatrix}}_{\frac{\omega^2}{c^2}\boldsymbol{\epsilon}}$$

$$= \begin{bmatrix} -k^2 + \frac{\omega^2}{c^2}\epsilon & 0 & 0 \\ 0 & -k^2 + \frac{\omega^2}{c^2}\epsilon & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{bmatrix} \tag{5.23}$$

Take determinant:

$$\det \ \mid D \mid = \left(-k^2 + \frac{\omega^2}{c^2}\epsilon\right)^2 \frac{\omega^2}{c^2}\epsilon = 0. \tag{5.24}$$

Two possible types of solution to this dispersion relation:

(A)

$$-k^2 + \frac{\omega^2}{c^2}\epsilon = 0. \tag{5.25}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\omega^2}{c^2}\epsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \qquad \Rightarrow E_z = 0. \tag{5.26}$$

Electric field is *transverse* ($\mathbf{E}.\mathbf{k} = 0$)

Phase velocity of the wave is

$$\frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} \tag{5.27}$$

This is just like a regular EM wave traveling in a medium with *refractive index*

$$N \equiv \frac{kc}{\omega} = \sqrt{\epsilon} \quad . \tag{5.28}$$

(B)

$$\frac{\omega^2}{c^2}\epsilon = 0 \qquad \text{i.e. } \epsilon = 0 \tag{5.29}$$

$$\Rightarrow \begin{pmatrix} D_{xx} & 0 & 0 \\ 0 & D_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \Rightarrow E_x = E_y = 0. \tag{5.30}$$

Electric Field is *Longitudinal* ($\mathbf{E} \wedge \mathbf{k} = 0$) $\quad \mathbf{E} \parallel \mathbf{k}$.

This has no obvious counterpart in optics etc. because ϵ is not usually zero. In plasmas $\epsilon = 0$ *is* a relevant solution. Plasmas can support longitudinal waves.

5.1.2 General Case (k in z-direction)

$$\mathbf{D} = \frac{\omega^2}{c^2}\begin{bmatrix} -N^2 + \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & -N^2 + \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \quad , \qquad \left(N^2 = \frac{k^2c^2}{\omega^2}\right) \tag{5.31}$$

When we take determinant we shall get a quadratic in N^2 (for given ω) provided $\boldsymbol{\epsilon}$ is not explicitly dependent on k. So for any ω there are two values of N^2. Two 'modes'. The polarization $\mathbf{E}$ of these modes will be in general partly longitudinal and partly transverse. *The point:* separation into distinct longitudinal and transverse modes is not possible in anisotropic media (e.g. plasma with B_o).

All we have said applies to general linear medium (crystal, glass, dielectric, plasma). Now we have to get the correct expression for $\boldsymbol{\sigma}$ and hence $\boldsymbol{\epsilon}$ by analysis of the plasma (fluid) equations.

5.2 High Frequency Plasma Conductivity

We want, now, to calculate the current for given (Fourier) electric field $\mathbf{E}(\mathbf{k}, \omega)$, to get the conductivity, $\boldsymbol{\sigma}$. It won't be the same as the DC conductivity which we calculated before (for collisions) because the *inertia* of the species will be important. In fact, provided

$$\omega \gg \bar{\nu}_{ei} \tag{5.32}$$

we can ignore collisions altogether. Do this for simplicity, although this approach can be generalized.

Also, under many circumstances we can ignore the pressure force $-\nabla p$. In general will be true if $\frac{\omega}{k} \gg v_{te,i}$ We take the plasma equilibrium to be at rest: $\mathbf{v}_o = 0$. This gives a manageable problem with wide applicability.

Approximations:

$$\begin{array}{cl} \text{Collisionless} & \bar{\nu}_{ei} = 0 \\ \text{`Cold Plasma'} & \nabla p = 0 \quad (e.g.\ T \simeq 0) \\ \text{Stationary Equil} & \mathbf{v}_o = 0 \end{array} \tag{5.33}$$

5.2.1 Zero B-field case

To start with take $\mathbf{B}_o = 0$: *Plasma isotropic* Momentum equation (for electrons first)

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v}\right] = nq\mathbf{E} \tag{5.34}$$

Notice the characteristic of the cold plasma approx. that we can cancel n from this equation and on linearizing get essentially the single particle equation.

$$m\frac{\partial \mathbf{v}_1}{\partial t} = q\mathbf{E} \qquad \text{(Drop the 1 suffix now).} \tag{5.35}$$

This can be solved for given ω as

$$\mathbf{v} = \frac{q}{-i\omega m}\mathbf{E} \tag{5.36}$$

and the current (due to this species, electrons) is

$$\mathbf{j} = nq\mathbf{v} = \frac{nq^2}{-i\omega m}\mathbf{E} \tag{5.37}$$

So the conductivity is

$$\sigma = i\frac{nq^2}{\omega m} \tag{5.38}$$

Hence dielectric constant is

$$\epsilon = 1 + \frac{i}{\omega\epsilon_o}\sigma = 1 - \left(\frac{nq^2}{m\epsilon_o}\right)\frac{1}{\omega^2} = 1 + \chi \tag{5.39}$$

Longitudinal Waves ($\mathbf{B}_o = 0$)

Dispersion relation we know is

$$\epsilon = 0 = 1 - \left(\frac{nq^2}{m\epsilon_o}\right)\frac{1}{\omega^2} \tag{5.40}$$

[Strictly, the ϵ we want here is the total ϵ including both electron and ion contributions to the conductivity. But

$$\frac{\sigma_e}{\sigma_i} \simeq \frac{m_i}{m_e} \quad (\text{for z } = 1) \tag{5.41}$$

so to a first approximation, ignore ion motions.]
Solution

$$\omega^2 = \left(\frac{n_e q_e^2}{m_e \epsilon_o}\right). \tag{5.42}$$

In this approx. longitudinal oscillations of the electron fluid have a single unique frequency:

$$\omega_p = \left(\frac{n_e e^2}{m_e \epsilon_o}\right)^{\frac{1}{2}}. \tag{5.43}$$

This is called the *'Plasma Frequency'* (more properly ω_{pe} the *'electron'* plasma frequency). If we allow for *ion motions* we get an ion conductivity

$$\sigma_i = \frac{i n_i q_i^2}{\omega m_i} \tag{5.44}$$

and hence

$$\begin{aligned} \epsilon_{\text{tot}} &= 1 + \frac{i}{\omega\epsilon_o}(\sigma_e + \sigma_i) = 1 - \left(\frac{n_e q_e^2}{\epsilon_o m_e} + \frac{n_i q_i^2}{\epsilon_o m_i}\right)\frac{1}{\omega^2} \\ &= 1 - \left(\omega_{pe}^2 + \omega_{pi}^2\right)/\omega^2 \end{aligned} \tag{5.45}$$

where

$$\omega_{pi} \equiv \left(\frac{n_i q_i^2}{\epsilon_o m_i}\right)^{\frac{1}{2}} \tag{5.46}$$

is the 'Ion Plasma Frequency'.

Simple Derivation of Plasma Oscillations

Take ions stationary; perturb a slab of plasma by shifting electrons a distance x. Charge built up is $n_e q x$ per unit area. Hence electric field generated

$$E = -\frac{n_e q_e x}{\epsilon_o} \tag{5.47}$$

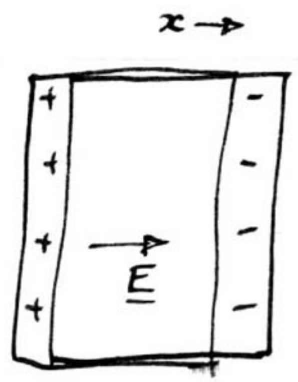

Figure 5.1: Slab derivation of plasma oscillations

Equation of motion of electrons

$$m_e \frac{dv}{dt} = -\frac{n_e q_e^2 x}{\epsilon_o}; \tag{5.48}$$

i.e.

$$\frac{d^2x}{dt^2} + \left(\frac{n_e q_e^2}{\epsilon_o m_e}\right) x = 0 \tag{5.49}$$

Simple harmonic oscillator with frequency

$$\omega_{pe} = \left(\frac{n_e q_e^2}{\epsilon_o m_e}\right)^{\frac{1}{2}} \qquad \text{Plasma Frequency.} \tag{5.50}$$

The Characteristic Frequency of Longitudinal Oscillations in a plasma. Notice

1. $\omega = \omega_p$ for all k in this approx.
2. Phase velocity $\frac{\omega}{k}$ can have any value.
3. Group velocity of wave, which is the velocity at which information/energy travel is

$$v_g = \frac{d\omega}{dk} = 0 \text{ !!} \tag{5.51}$$

In a way, these oscillations can hardly be thought of as a 'proper' wave because they do not transport energy or information. (In Cold Plasma Limit). [Nevertheless they do emerge from the wave analysis and with less restrictive approxs do have finite v_g.]

Transverse Waves ($B_o = 0$)

Dispersion relation:

$$-k^2 + \frac{\omega^2}{c^2}\epsilon = 0 \tag{5.52}$$

or

$$N^2 \equiv \frac{k^2c^2}{\omega^2} = \epsilon = 1 - \left(\omega_{pe}^2 + \omega_{pi}^2\right)/\omega^2$$
$$\simeq 1 - \omega_{pe}^2/\omega^2 \tag{5.53}$$

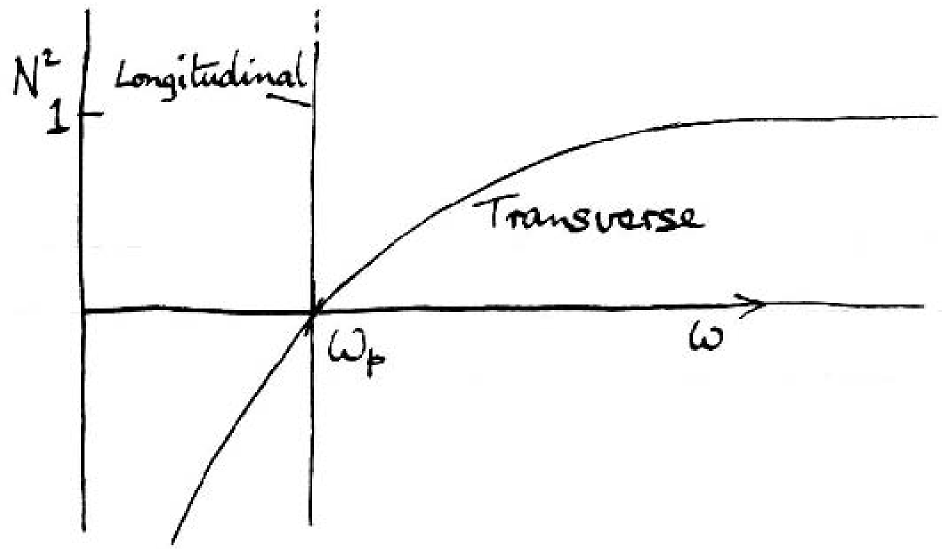

Figure 5.2: Unmagnetized plasma transverse wave.

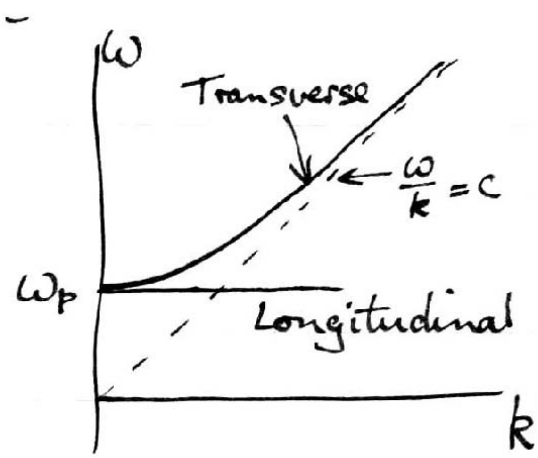

Figure 5.3: Alternative dispersion plot.

Alternative expression:

$$-k^2 + \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} = 0 \tag{5.54}$$

which implies

$$\omega^2 = \omega_p^2 + k^2c^2 \tag{5.55}$$

$$\omega = \left(\omega_p^2 + k^2c^2\right)^{\frac{1}{2}} . \tag{5.56}$$

5.2.2 Meaning of Negative N^2: Cut Off

When $N^2 < 0$ (for $\omega < \omega_p$) this means N is pure imaginary and hence so is k for real ω. Thus the wave we have found goes like

$$\exp\{\pm \mid k \mid x - i\omega t\} \tag{5.57}$$

i.e. its space dependence is exponential *not oscillatory*. Such a wave is said to be *'Evanescent'* or 'Cut Off'. It does not truly propagate through the medium but just damps exponentially.

Example:

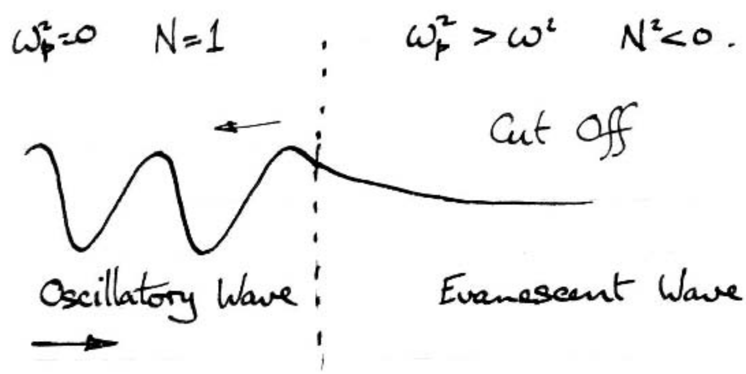

Figure 5.4: Wave behaviour at cut-off.

A wave incident on a plasma with $\omega_p^2 > \omega^2$ is simply reflected, no energy is transmitted through the plasma.

5.3 Cold Plasma Waves (Magnetized Plasma)

Objective: calculate $\boldsymbol{\epsilon}$,$\mathbf{D}, \mathbf{k}(\omega)$, using known plasma equations.

Approximation: Ignore thermal motion of particles.

Applicability: Most situations where (1) plasma pressure and (2) absorption are negligible. Generally requires wave phase velocity $\gg v_{\text{thermal}}$.

5.3.1 Derivation of Dispersion Relation

Can "derive" the cold plasma approx from fluid plasma equations. Simpler just to say that all particles (of a specific species) just move together obeying Newton's 2nd law:

$$m\frac{\partial \mathbf{v}}{\partial t} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{5.58}$$

Take the background plasma to have $\mathbf{E}_0 = 0$, $\mathbf{B} = \mathbf{B}_0$ and zero velocity. Then all motion is due to the wave and also the wave's magnetic field can be ignored provided the particle speed stays small. (This is a linearization).

$$m\frac{\partial \mathbf{v}}{\partial t} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}_0), \tag{5.59}$$

where $\mathbf{v}$, $\mathbf{E} \propto \exp i(\mathbf{k}.\mathbf{x} - \omega t)$ are wave quantities.

Substitute $\frac{\partial}{\partial t} \to -i\omega$ and write out equations. Choose axes such that $\mathbf{B}_0 = B_0(0, 0, 1)$.

$$\begin{aligned} -i\omega m v_x &= q(E_x + v_y B_0) \\ -i\omega m v_y &= q(E_y - v_x B_0) \\ -i\omega m v_z &= qE_z \end{aligned} \tag{5.60}$$

Solve for $\mathbf{v}$ in terms of $\mathbf{E}$.

$$\begin{aligned} v_x &= \frac{q}{m}\left(\frac{i\omega E_x - \Omega E_y}{\omega^2 - \Omega^2}\right) \\ v_y &= \frac{q}{m}\left(\frac{\Omega E_x + i\omega E_y}{\omega^2 - \Omega^2}\right) \\ v_z &= \frac{q}{m}\frac{i}{\omega}E_z \end{aligned} \tag{5.61}$$

where $\Omega = \frac{qB_0}{m}$ is the gyrofrequency but its sign is that of the charge on the particle species under consideration.

Since the current is $\mathbf{j} = q\mathbf{v}n = \boldsymbol{\sigma}.\mathbf{E}$ we can identify the conductivity tensor for the species (j) as:

$$\boldsymbol{\sigma}_j = \begin{bmatrix} \frac{q_j^2 n_j}{m_j}\frac{i\omega}{\omega^2 - \Omega_j^2} & -\frac{q_j^2 n_j}{m_j}\frac{\Omega_j}{\omega^2 - \Omega_j^2} & 0 \\ \frac{q_j^2 n_j}{m_j}\frac{\Omega_j}{\omega^2 - \Omega_j^2} & \frac{q_j^2 n_j}{m_j}\frac{i\omega}{\omega^2 - \Omega_j^2} & 0 \\ 0 & 0 & \frac{iq_j^2}{m}\frac{n_j}{\omega} \end{bmatrix} \tag{5.62}$$

The total conductivity, due to all species, is the sum of the conductivities for each

$$\boldsymbol{\sigma} = \sum_j \boldsymbol{\sigma}_j \tag{5.63}$$

So

$$\sigma_{xx} = \sigma_{yy} = \sum_j \frac{q_1^2 n_j}{m_j}\frac{i\omega}{\omega^2 - \Omega_j^2} \tag{5.64}$$

$$\sigma_{xy} = -\sigma_{yx} = -\sum_j \frac{q_j^2 n_j}{m_j} \frac{\Omega_j}{\omega^2 - \Omega_j^2} \tag{5.65}$$

$$\sigma_{zz} = \sum_j \frac{q_j^2 n_j}{m_j} \frac{i}{\omega} \tag{5.66}$$

Susceptibility $\chi = \frac{1}{-i\omega\epsilon_o}\boldsymbol{\sigma}$.

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \tag{5.67}$$

where

$$\epsilon_{xx} = \epsilon_{yy} = S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \tag{5.68}$$

$$i\epsilon_{xy} = -i\epsilon_{yx} = D = \sum_j \frac{\Omega_j}{\omega} \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \tag{5.69}$$

$$\epsilon_{zz} = P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \tag{5.70}$$

and

$$\omega_{pj}^2 \equiv \frac{q_j^2 n_j}{\epsilon_o m_j} \tag{5.71}$$

is the "plasma frequency" for that species.

S & D stand for "Sum" and "Difference":

$$S = \frac{1}{2}(R + L) \qquad D = \frac{1}{2}(R - L) \tag{5.72}$$

where R & L stand for "Right-hand" and "Left-hand" and are:

$$R = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\,(\omega + \Omega_j)} \quad , \quad L = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\,(\omega - \Omega_j)} \tag{5.73}$$

The R & L terms arise in a derivation based on expressing the field in terms of rotating polarizations (right & left) rather than the direct Cartesian approach.

We now have the dielectric tensor from which to obtain the dispersion relation and solve it to get $\mathbf{k}(\omega)$ and the polarization. Notice, first, that $\boldsymbol{\epsilon}$ is indeed independent of $\mathbf{k}$ so the dispersion relation (for given ω) is a quadratic in N^2 (or k^2).

Choose convenient axes such that $k_y = N_y = 0$. Let θ be angle between $\mathbf{k}$ and $\mathbf{B}_0$ so that

$$N_z = N\cos\theta \ , \qquad N_x = N\sin\theta \ . \tag{5.74}$$

Then

$$\mathbf{D} = \begin{bmatrix} -N^2\cos^2\theta + S & -iD & N^2\sin\theta\cos\theta \\ +iD & -N^2 + S & 0 \\ N^2\sin\theta\cos\theta & 0 & -N^2\sin^2\theta + P \end{bmatrix} \tag{5.75}$$

and

$$\| \mathbf{D} \| = AN^4 - BN^2 + C \tag{5.76}$$

where

$$A \equiv S\sin^2\theta + P\cos^2\theta \tag{5.77}$$

$$B \equiv RL\sin^2\theta + PS(1+\cos^2\theta) \tag{5.78}$$

$$C \equiv PRL \tag{5.79}$$

Solutions are

$$N^2 = \frac{B \pm F}{2A}, \tag{5.80}$$

where the discriminant, F, is given by

$$F^2 = (RL - PS)^2 \sin^4\theta + 4P^2D^2\cos^2\theta \tag{5.81}$$

after some algebra. This is often, for historical reasons, written in the equivalent form (called the Appleton-Hartree dispersion relation)

$$N^2 = 1 - \frac{2(A - B + C)}{2A - B \pm F} \tag{5.82}$$

The quantity F^2 is generally $+ve$, so N^2 is real $\Rightarrow$ "propagating" or "evanescent" *no* wave absorption for cold plasma.

Solution can also be written

$$\tan^2\theta = -\frac{P\,(N^2 - R)\,(N^2 - L)}{(SN^2 - RL)\,(N^2 - P)} \tag{5.83}$$

This compact form makes it easy to identify the dispersion relation at $\theta = 0$ & $\frac{\pi}{2}$ i.e. parallel and perpendicular propagation $\tan\theta = 0, \infty$.

Parallel: $P = 0$, $N^2 = R$ $N^2 = L$

Perp: $N^2 = \frac{RL}{S}$ $N^2 = P$.

Example: Right-hand wave

$N^2 = R$. (Single Ion Species).

$$N^2 = 1 - \frac{\omega_{pe}^2}{\omega\,(\omega - |\Omega_e|)} - \frac{\omega_{pi}^2}{\omega\,(\omega + |\Omega_i|)} \tag{5.84}$$

This has a wave resonance $N^2 \to \infty$ at $\omega = |\Omega_e|$, only. Right-hand wave also has a cutoff at $R = 0$, whose solution proves to be

$$\omega = \omega_R = \frac{|\Omega_e| - |\Omega_i|}{2} + \left[\left(\frac{|\Omega_e| + |\Omega_i|}{2}\right)^2 + \omega_{pe}^2 + \omega_{pi}^2\right]^{1/2} \tag{5.85}$$

Since $m_i \gg m_e$ this can be approximated as:

$$\omega_R \simeq \frac{|\Omega_e|}{2}\left\{1+\left(1+4\frac{\omega_{pe}^2}{|\Omega_e|^2}\right)^{\frac{1}{2}}\right\} \tag{5.86}$$

This is always above $|\Omega_e|$.

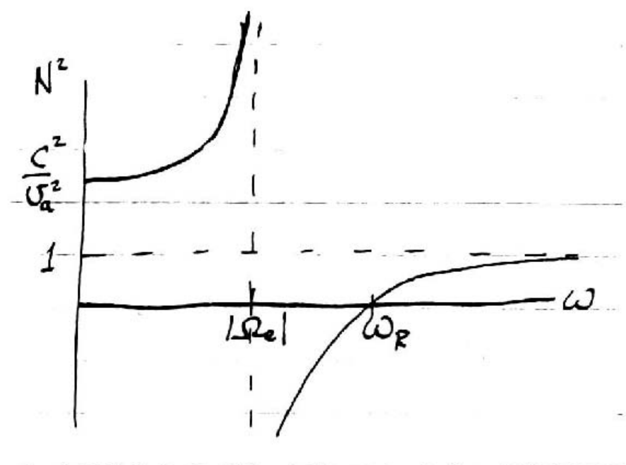

Figure 5.5: The form of the dispersion relation for RH wave.

One can similarly investigate LH wave and perp propagating waves. The resulting wave resonances and cut-offs depend only upon 2 properties (for specified ion mass) (1) Density $\leftrightarrow \omega_{pe}^2$ (2) Magnetic Field $\leftrightarrow |\Omega_e|$. [Ion values ω_{pi}, $|\Omega_i|$ are got by $\frac{m_i}{m_e}$ factors.]

These resonances and cutoffs are often plotted on a 2-D plane $\frac{|\Omega_e|}{\omega}$, $\frac{\omega_p^2}{\omega^2}$ $(\propto B, n)$ called the C M A Diagram.

We don't have time for it here.

5.3.2 Hybrid Resonances Perpendicular Propagation

"Extraordinary" wave $\quad N^2 = \frac{RL}{S}$

$$N^2 = \frac{\left[(\omega+\Omega_e)(\omega+\Omega_i) - \frac{\omega_{pe}^2}{\omega}(\omega+\Omega_i) - \frac{\omega_{pi}^2}{\omega}(\omega+\Omega_e)\right]\left[(\omega-\Omega_e)(\omega-\Omega_i) - \frac{\omega_{pe}^2}{\omega}(\omega-\Omega_i)\ldots\right]}{(\omega^2-\Omega_e^2)(\omega^2-\Omega_i^2) - \omega_{pe}^2(\omega^2-\Omega_i^2) - \omega_{pi}^2(\omega^2-\Omega_e^2)} \tag{5.87}$$

Resonance is where denominator = 0. Solve the quadratic in ω^2 and one gets

$$\omega^2 = \frac{\omega_{pe}^2+\Omega_e^2+\omega_{pi}^2+\Omega_i^2}{2} \pm \sqrt{\left(\frac{\omega_{pe}^2+\Omega_e^2-\omega_{pi}^2-\Omega_i^2}{2}\right)^2+\omega_{pe}^2\omega_{pi}^2} \tag{5.88}$$

Neglecting terms of order $\frac{m_e}{m_i}$ (e.g. $\frac{\omega_{pi}^2}{\omega_{pe}^2}$) one gets solutions

$$\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2 \qquad \text{Upper Hybrid Resonance.} \tag{5.89}$$

$$\omega_{LH}^2 = \frac{\Omega_e^2 \omega_{pi}^2}{\Omega_e^2 + \omega_{pe}^2} \qquad \text{Lower Hybrid Resonance..} \tag{5.90}$$

At very high density, $\omega_{pe}^2 \gg \Omega_e^2$

$$\omega_{LH}^2 \simeq |\Omega_e||\Omega_i| \tag{5.91}$$

geometric mean of cyclotron frequencies.
At very low density, $\omega_{pe}^2 \ll \Omega_e^2$

$$\omega_{LH}^2 \simeq \omega_{pi}^2 \tag{5.92}$$

ion plasma frequency
Usually in tokamaks $\omega_{pe}^2 \sim \Omega_e^2$. Intermediate.

Summary Graph ($\Omega > \omega_p$)

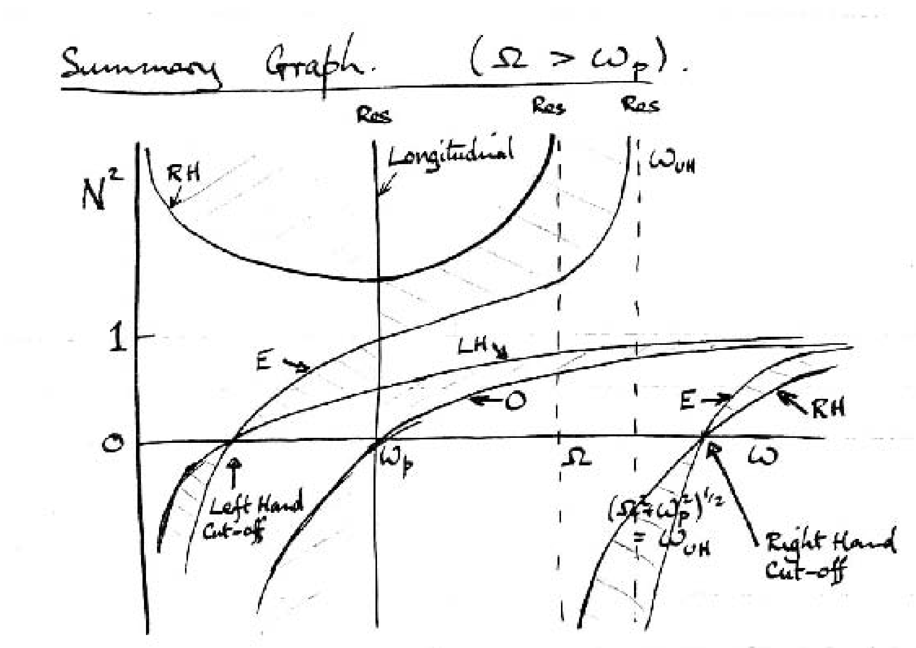

Figure 5.6: Summary of magnetized dispersion relation

Cut-offs are where $N^2 = 0$.
Resonances are where $N^2 \to \infty$.
Intermediate angles of propagation have refractive indices between the $\theta = 0$, $\frac{\pi}{2}$ lines, in the shaded areas.

5.3.3 Whistlers

(Ref. R.A. Helliwell, "Whistlers & Related Ionospheric Phenomena," Stanford UP 1965.)
For $N^2 \gg 1$ the right hand wave can be written

$$N^2 \simeq \frac{-\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} \quad , \qquad (N = kc/\omega) \tag{5.93}$$

Group velocity is

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \left[\frac{d}{d\omega}\left(\frac{N\omega}{c}\right)\right]^{-1}. \tag{5.94}$$

Then since

$$N = \frac{\omega_p}{\omega^{\frac{1}{2}}(|\Omega_e| - \omega)^{\frac{1}{2}}} \quad , \tag{5.95}$$

we have

$$\begin{aligned} \frac{d}{d\omega}(N\omega) &= \frac{d}{d\omega}\frac{\omega_p \omega^{\frac{1}{2}}}{(|\Omega_e| - \omega)^{\frac{1}{2}}} = \omega_p \left\{ \frac{\frac{1}{2}}{\omega^{\frac{1}{2}}(|\Omega_e| - \omega)^{\frac{1}{2}}} + \frac{\frac{1}{2}\omega^{\frac{1}{2}}}{(|\Omega_e| - \omega)^{\frac{3}{2}}} \right\} \\ &= \frac{\omega_p/2}{(|\Omega_e| - \omega)^{\frac{3}{2}}\omega^{\frac{1}{2}}}\{(|\Omega_e| - \omega) + \omega\} \\ &= \frac{\omega_p|\Omega_e|/2}{(|\Omega_e| - \omega)^{\frac{3}{2}}\omega^{\frac{1}{2}}} \end{aligned} \tag{5.96}$$

Thus

$$v_g = \frac{c\,2\,(|\Omega_e| - \omega)^{\frac{3}{2}}\omega^{\frac{1}{2}}}{\omega_p|\Omega_e|} \tag{5.97}$$

Group Delay is

$$\frac{L}{v_g} \propto \frac{1}{\omega^{\frac{1}{2}}(|\Omega_e| - \omega)^{\frac{3}{2}}} \propto \frac{1}{\left(\frac{\omega}{|\Omega_e|}\right)^{\frac{1}{2}}\left(1 - \frac{\omega}{|\Omega_e|}\right)^{\frac{3}{2}}} \tag{5.98}$$

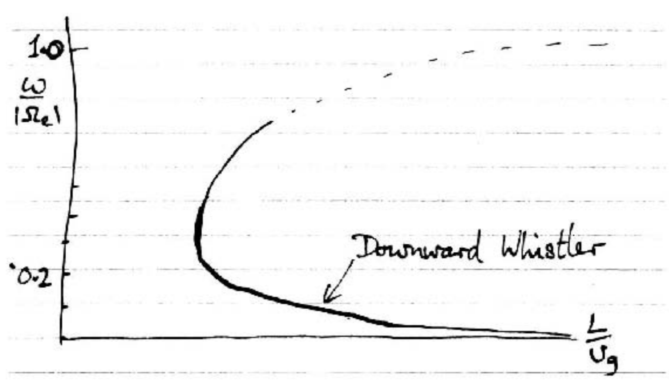

Figure 5.7: Whistler delay plot

Plot with $\frac{L}{v_g}$ as x-axis.

Resulting form explains downward whistle.

Lightning strike $\sim \delta$-function excites all frequencies.

Lower ones arrive later.

Examples of actual whistler sounds can be obtained from `http://www-istp.gsfc.nasa.gov/istp/polar/polar_pwi_sounds.html`.

5.4 Thermal Effects on Plasma Waves

The cold plasma approx is only good for high frequency, $N^2 \sim 1$ waves. If ω is low or $N^2 \gg 1$ one may have to consider thermal effects. From the fluid viewpoint, this means *pressure*. Write down the momentum equation. (We shall go back to $B_0 = 0$) linearized

$$mn\frac{\partial \mathbf{v}_1}{\partial t} = nq\mathbf{E}_1 - \nabla p_1 \quad ; \tag{5.99}$$

remember these are the perturbations:

$$p = p_0 + p_1 \quad . \tag{5.100}$$

Fourier Analyse (drop 1's)

$$mn(-i\omega)\mathbf{v} = nq\mathbf{E} - i\mathbf{k}p \tag{5.101}$$

The *key question*: how to relate p to $\mathbf{v}$

Answer: *Equation of state + Continuity*

State

$$pn^{-\gamma} = \text{const.} \Rightarrow (p_0 + p_1)(n_0 + n_1)^{-\gamma} = p_0 n_0^{-\gamma} \tag{5.102}$$

Use Taylor Expansion

$$(p_0 + p_1)(n_0 + n_1)^{-\gamma} \simeq p_0 n_0^{-\gamma}\left[1 + \frac{p_1}{p_0} - \gamma\frac{n_1}{n_0}\right] \tag{5.103}$$

Hence

$$\frac{p_1}{p_0} = \gamma\frac{n_1}{n_0} \tag{5.104}$$

Continuity

$$\frac{\partial n}{\partial t} + \nabla.(n\mathbf{v}) = 0 \tag{5.105}$$

Linearise:

$$\frac{\partial n_1}{\partial t} + \nabla.(n_0\mathbf{v}_1) = 0 \quad \Rightarrow \quad \frac{\partial n}{\partial t} + n_0\nabla.\mathbf{v} = 0 \tag{5.106}$$

Fourier Transform

$$-i\omega n_1 + n_0 i\mathbf{k}.\mathbf{v}_1 = 0 \tag{5.107}$$

i.e.

$$n_1 = n_0 \frac{\mathbf{k}.\mathbf{v}}{\omega} \tag{5.108}$$

Combine State & Continuity

$$p_1 = p_0 \gamma \frac{n_1}{n_0} = p_0 \gamma \frac{n_0 \frac{\mathbf{k}.\mathbf{v}}{\omega}}{n_o} = p_0 \gamma \frac{\mathbf{k}.\mathbf{v}}{\omega} \tag{5.109}$$

Hence Momentum becomes

$$mn\left(-i\omega\right)\mathbf{v} = nq\mathbf{E} - \frac{ikp_0\gamma}{\omega}\mathbf{k}.\mathbf{v} \tag{5.110}$$

Notice *Transverse waves* have $\mathbf{k}.\mathbf{v} = 0$; so they are *unaffected by pressure*.

Therefore we need only consider the longitudinal wave. However, for consistency let us proceed as before to get the dielectric tensor etc.

Choose axes such that $\mathbf{k} = k\hat{\mathbf{e}}_z$ then obviously:

$$v_x = \frac{iq}{\omega m} E_x \qquad v_y = \frac{iq}{\omega m} E_y \tag{5.111}$$

$$v_z = \frac{q}{m} \frac{E_z}{-i\omega + (ik^2 \gamma p_0 / mn\omega)} \tag{5.112}$$

Hence

$$\boldsymbol{\sigma} = \frac{inq^2}{\omega m} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1 - \frac{k^2 p_0 \gamma}{mn\omega^2}} \end{bmatrix} \tag{5.113}$$

$$\boldsymbol{\epsilon} = 1 + \frac{i\boldsymbol{\sigma}}{\epsilon_0 \omega} = \begin{bmatrix} 1 - \frac{\omega_p^2}{\omega^2} & 0 & 0 \\ 0 & 1 - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 - \frac{w_p^2}{\omega^2 - k^2 \frac{p_0 \gamma}{mn}} \end{bmatrix} \tag{5.114}$$

(Taking account only of 1 species, electrons, for now.)

We have confirmed the previous comment that the transverse waves (E_x, E_y) are unaffected. The longitudinal wave *is*. Notice that $\boldsymbol{\epsilon}$ now depends on $\mathbf{k}$ as well as ω. This is called *'spatial dispersion'*.

For completeness, note that the dielectric tensor can be expressed in general tensor notation as

$$\begin{aligned} \boldsymbol{\epsilon} &= \mathbf{1} - \frac{\omega_p^2}{\omega^2} \left(\mathbf{1} + \mathbf{kk} \left[\frac{1}{1 - \frac{k^2}{\omega^2} \frac{p_0 \gamma}{mn}} - 1 \right] \right) \\ &= \mathbf{1} - \frac{\omega_p^2}{\omega^2} \left(\mathbf{1} + \mathbf{kk} \frac{1}{\frac{w^2 mn}{k^2 p_0 \gamma} - 1} \right) \end{aligned} \tag{5.115}$$

This form shows isotropy with respect to the medium: there is no preferred direction in space for the wave vector $\mathbf{k}$.

But once $\mathbf{k}$ is chosen, $\boldsymbol{\epsilon}$ is not isotropic. The direction of $\mathbf{k}$ becomes a special direction.

Longitudinal Waves: dispersion relation is

$$\epsilon_{zz} = 0 \qquad \text{(as before)} \tag{5.116}$$

which is

$$1 - \frac{\omega_p^2}{\omega^2 - \frac{k^2 p_0 \gamma}{mn}} = 0 \quad . \tag{5.117}$$

or

$$\omega^2 = \omega_p^2 + k^2 \frac{p_0 \gamma}{mn} \tag{5.118}$$

Recall $p_0 = n_0 T = nT$; so this is usually written:

$$\omega^2 = \omega_p^2 + k^2 \frac{\gamma T}{m} = \omega_p^2 + k^2 \gamma v_t^2 \tag{5.119}$$

[The appropriate value of γ to take is 1 dimensional adiabatic i.e. $\gamma = 3$. This seems plausible since the electron motion is 1-d (along k) and may be demonstrated more rigorously by kinetic theory.]

The above dispersion relation is called the *Bohm-Gross* formula for *electron plasma waves*. Notice the group velocity:

$$v_g = \frac{d\omega}{dk} = \frac{1}{2\omega}\frac{d\omega^2}{dk} = \frac{\gamma k v_t^2}{\left(\omega_p^2 + \gamma k^2 v_t^2\right)^{\frac{1}{2}}} \neq 0. \tag{5.120}$$

and for $k v_t > \omega_p$ this tends to $\gamma^{\frac{1}{2}} v_t$. In this limit energy travels at the electron thermal speed.

5.4.1 Refractive Index Plot

Bohm Gross electron plasma waves:

$$N^2 = \frac{c^2}{\gamma_e \mathbf{v}_{te}^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.121}$$

Transverse electromagnetic waves:

$$N^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.122}$$

These have just the same shape except the electron plasma waves have much larger vertical scale:

On the E-M wave scale, the plasma wave curve is nearly vertical. In the cold plasma it was *exactly* vertical.

We have relaxed the Cold Plasma approximation.

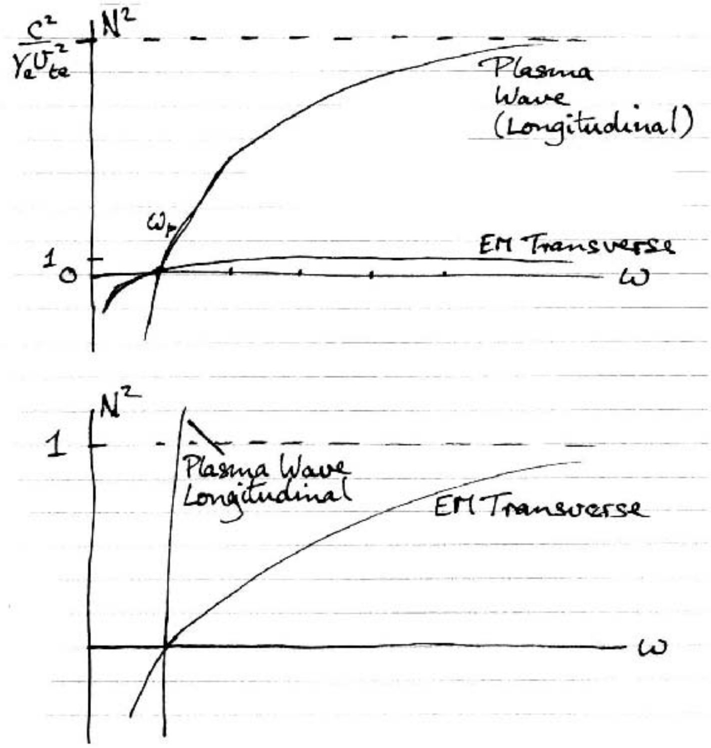

Figure 5.8: Refractive Index Plot. Top plot on the scale of the Bohm-Gross Plasma waves. Bottom plot, on the scale of the E-M transverse waves

5.4.2 Including the ion response

As an example of the different things which can occur when ions are allowed to move include longitudinal ion response:

$$0 = \epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} \tag{5.123}$$

This is now a quadratic equation for ω^2 so there are *two* solutions possible for a given ω. One will be in the vicinity of the electron plasma wave solution and the inclusion of ω_{pi}^2 which is $\ll \omega_{pe}^2$ will give a small correction.

Second solution will be where the third term is same magnitude as second (both will be $\gg 1$). This will be at low frequency. So we may write the dispersion relation approximately as:

$$-\frac{\omega_{pi}^2}{-\frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} = 0 \tag{5.124}$$

i.e.

$$\omega^2 = \frac{k^2 p_i \gamma_i}{m_i n_i} + \frac{\omega_{pi}^2}{\omega_{pe}^2} \frac{k^2 p_e \gamma_e}{m_e n_e}$$

$$
\begin{aligned}
&= k^2\left[\left(\frac{\gamma_i p_i}{n_i}+\frac{\gamma_e p_e}{n_e}\right)\frac{1}{m_i}\right] \\
&= k^2\left[\frac{\gamma_i T_i+\gamma_e T_e}{m_i}\right]
\end{aligned}
\tag{5.125}
$$

[In this case the electrons have time to stream through the wave in 1 oscillation so they tend to be isothermal: i.e. $\gamma_e = 1$. What to take for γ_i is less clear, and less important because kinetic theory shows that these waves we have just found are strongly damped unless $T_i \ll T_e$.]

These are *'ion-acoustic'* or 'ion-sound' waves

$$
\frac{\omega^2}{k^2} = c_s^2 \tag{5.126}
$$

c_s is the sound speed

$$
c_s^2 = \frac{\gamma_i T_i + T_e}{m_i} \simeq \frac{T_e}{m_i} \tag{5.127}
$$

Approximately non-dispersive waves with phase velocity c_s.

5.5 Electrostatic Approximation for (Plasma) Waves

The dispersion relation is written generally as

$$
\mathbf{N}\wedge(\mathbf{N}\wedge\mathbf{E}) + \boldsymbol{\epsilon}.\mathbf{E} = \mathbf{N}(\mathbf{N}.\mathbf{E}) - N^2\mathbf{E} + \boldsymbol{\epsilon}.\mathbf{E} = 0 \tag{5.128}
$$

Consider $\mathbf{E}$ to be expressible as longitudinal and transverse components $\mathbf{E}_\ell$, $\mathbf{E}_t$ such that $\mathbf{N}\wedge\mathbf{E}_\ell = 0$, $\mathbf{N}.\mathbf{E}_t = 0$. Then the dispersion relation can be written

$$
\mathbf{N}\left(\mathbf{N}.\mathbf{E}_\ell\right) - N^2\left(\mathbf{E}_\ell + \mathbf{E}_t\right) + \boldsymbol{\epsilon}.\left(\mathbf{E}_\ell + \mathbf{E}_t\right) = -N^2\mathbf{E}_t + \boldsymbol{\epsilon}.\mathbf{E}_t + \boldsymbol{\epsilon}.E_\ell = 0 \tag{5.129}
$$

or

$$
\left(N^2 - \boldsymbol{\epsilon}\right).\mathbf{E}_t = \boldsymbol{\epsilon}.\mathbf{E}_\ell \tag{5.130}
$$

Now the electric field can always be written as the sum of a curl-free component plus a divergenceless component, e.g. conventionally

$$
\mathbf{E} = \underbrace{-\nabla\phi}_{\substack{Curl-free \\ Electrostatic}} + \underbrace{\dot{\mathbf{A}}}_{\substack{Divergence-free \\ Electromagnetic}} \tag{5.131}
$$

and these may be termed electrostatic and electromagnetic parts of the field.

For a plane wave, these two parts are clearly the same as the longitudinal and transverse parts because

$$
-\nabla\phi = -i\mathbf{k}\phi \quad \text{is longitudinal} \tag{5.132}
$$

and if $\nabla.\dot{\mathbf{A}} = 0$ (because $\nabla.\mathbf{A} = 0$ (w.l.o.g.)) then $\mathbf{k}.\dot{\mathbf{A}} = 0$ so $\dot{\mathbf{A}}$ is transverse.

'Electrostatic' waves are those that are describable by the electrostatic part of the electric field, which is the longitudinal part: $|E_\ell| \gg |E_t|$.

If we simply say $\mathbf{E}_t = 0$ then the dispersion relation becomes $\boldsymbol{\epsilon}.\mathbf{E}_\ell = 0$. This is *not* the most general dispersion relation for electrostatic waves. It is too restrictive. In general, there is a more significant way in which to get solutions where $|E_\ell| \gg |E_t|$. It is for N^2 to be very large compared to all the components of $\boldsymbol{\epsilon} : N^2 \gg \| \epsilon \|$.

If this is the case, then the dispersion relation is approximately

$$N^2 \mathbf{E}_t = \boldsymbol{\epsilon}.\mathbf{E}_\ell \quad ; \tag{5.133}$$

$\mathbf{E}_t$ is small but not zero.

We can then annihilate the $\mathbf{E}_t$ term by taking the $\mathbf{N}$ component of this equation; leaving

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{E}_\ell = (\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N})\, E_\ell = 0 \quad : \quad \mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k} = 0\ . \tag{5.134}$$

When the medium is isotropic there is no relevant difference between the electrostatic dispersion relation:

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N} = 0 \tag{5.135}$$

and the purely longitudinal case $\boldsymbol{\epsilon}.\mathbf{N} = \mathbf{0}$. If we choose axes such that $\mathbf{N}$ is along $\hat{\mathbf{z}}$, then the medium's isotropy ensures the off-diagonal components of $\boldsymbol{\epsilon}$ are zero so $\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N} = 0$ requires $\epsilon_{zz} = 0 \Rightarrow \boldsymbol{\epsilon}.\mathbf{N} = \mathbf{0}$. However if the medium is *not* isotropic, then even if

$$\mathbf{N}.\boldsymbol{\epsilon}.\mathbf{N} \left(= N^2 \epsilon_{zz} \right) = 0 \tag{5.136}$$

there may be off-diagonal terms of $\boldsymbol{\epsilon}$ that make

$$\boldsymbol{\epsilon}.\mathbf{N} \neq 0 \tag{5.137}$$

In other words, in an anisotropic medium (for example a magnetized plasma) the electrostatic approximation can give waves that have non-zero transverse electric field (of order $||\boldsymbol{\epsilon}||/N^2$ times E_ℓ) even though the waves are describable in terms of a scalar potential.

To approach this more directly, from Maxwell's equations, applied to a dielectric medium of dielectric tensor $\boldsymbol{\epsilon}$, the electrostatic part of the electric field is derived from the electric displacement

$$\nabla.\mathbf{D} = \nabla.\left(\epsilon_0 \boldsymbol{\epsilon}.\mathbf{E}\right) = \rho = 0 \quad \text{(no free charges)} \tag{5.138}$$

So for plane waves $0 = \mathbf{k}.\mathbf{D} = \mathbf{k}.\boldsymbol{\epsilon}.\mathbf{E} = i\mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k}\phi$.

The electric displacement, $\mathbf{D}$, is purely transverse (not zero) but the electric field, $\mathbf{E}$ then gives rise to an electromagnetic field via $\nabla \wedge \mathbf{H} = \partial \mathbf{D}/\partial t$. If $N^2 \gg \| \epsilon \|$ then this magnetic (inductive) component can be considered as a benign passive coupling to the electrostatic wave.

In summary, the electrostatic dispersion relation is $\mathbf{k}.\boldsymbol{\epsilon}.\mathbf{k} = 0$, or in coordinates where $\mathbf{k}$ is in the z-direction, $\epsilon_{zz} = 0$.

5.6 Simple Example of MHD Dynamics: Alfven Waves

Ignore Pressure & Resistance.

$$\rho\frac{D\mathbf{V}}{Dt} = \mathbf{j}\wedge\mathbf{B} \tag{5.139}$$

$$\mathbf{E}+\mathbf{V}\wedge\mathbf{B}=0 \tag{5.140}$$

Linearize:

$$\mathbf{V}=\mathbf{V}_1,\quad \mathbf{B}=\mathbf{B}_0+\mathbf{B}_1\ (\mathbf{B}_0\text{ uniform}),\quad \mathbf{j}=\mathbf{j}_1. \tag{5.141}$$

$$\rho\frac{\partial\mathbf{V}}{\partial t} = \mathbf{j}\wedge\mathbf{B}_0 \tag{5.142}$$

$$\mathbf{E}+\mathbf{V}\wedge\mathbf{B}_0=0 \tag{5.143}$$

Fourier Transform:

$$\rho(-i\omega)\mathbf{V} = \mathbf{j}\wedge\mathbf{B}_0 \tag{5.144}$$

$$\mathbf{E}+\mathbf{V}\wedge\mathbf{B}_0=0 \tag{5.145}$$

Eliminate V by taking 5.144 $\wedge\mathbf{B}_0$ and substituting from 5.145.

$$\mathbf{E}+\frac{1}{-i\omega\rho}\left(\mathbf{j}\wedge\mathbf{B}_0\right)\wedge\mathbf{B}_0=0 \tag{5.146}$$

or

$$\mathbf{E}=-\frac{1}{-i\omega\rho}\{(\mathbf{j}.\mathbf{B}_0)\,\mathbf{B}_0-B_0^2\mathbf{j}\}=\frac{B_0^2}{-i\omega\rho}\mathbf{j}_\perp \tag{5.147}$$

So conductivity tensor can be written (z in B direction).

$$\boldsymbol{\sigma}=\frac{-i\omega\rho}{B_0^2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&\infty\end{bmatrix} \tag{5.148}$$

where ∞ implies that $E_\parallel=0$ (because of Ohm's law). Hence Dielectric Tensor

$$\boldsymbol{\epsilon}=1+\frac{\boldsymbol{\sigma}}{-i\omega\epsilon_0}=\left(1+\frac{\rho}{\epsilon_0B^2}\right)\begin{bmatrix}1&0&0\\0&1&0\\0&0&\infty\end{bmatrix}. \tag{5.149}$$

Dispersion tensor in general is:

$$\mathbf{D}=\frac{\omega^2}{c^2}\left[\mathbf{NN}-N^2+\boldsymbol{\epsilon}\right] \tag{5.150}$$

Dispersion Relation taking $N_\perp=N_x$, $N_y=0$

$$|\mathbf{D}|=\left|\begin{bmatrix}-N_\parallel^2+1+\frac{\rho}{\epsilon_0B^2}&0&N_\perp N_\parallel\\0&-N_\parallel^2-N_\perp^2+1+\frac{\rho}{\epsilon_0B^2}&0\\N_\perp N_\parallel&0&\infty\end{bmatrix}\right|=0 \tag{5.151}$$

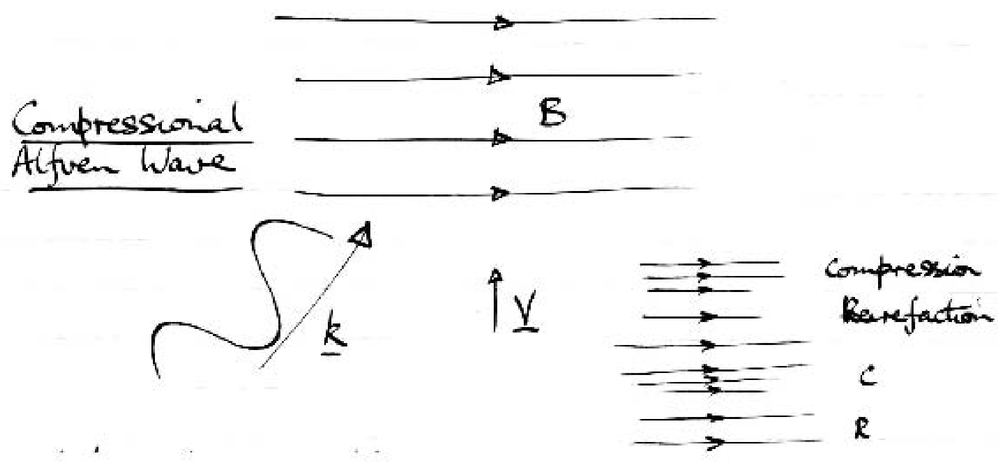

Figure 5.9: Compressional Alfven Wave. Works by magnetic pressure (primarily).

Meaning of ∞ is that the cofactor must be zero i.e.

$$\left(-N_{\parallel}^2 + 1 + \frac{\rho}{\epsilon_0 B^2}\right)\left(-N^2 + 1 + \frac{\rho}{\epsilon_0 B^2}\right) = 0 \tag{5.152}$$

The 1's here come from Maxwell displacement current and are usually negligible ($N_{\perp}^2 \gg 1$). So final waves are

1. $N^2 = \frac{\rho}{\epsilon_0 B^2} \Rightarrow$ Non-dispersive wave with phase and group velocities

$$v_p = v_g = \frac{c}{N} = \left(\frac{c^2 \epsilon_0 B^2}{\rho}\right)^{\frac{1}{2}} = \left[\frac{B^2}{\mu_0 \rho}\right]^{\frac{1}{2}} \tag{5.153}$$

where we call

$$\left[\frac{B^2}{\mu_0 \rho}\right]^{\frac{1}{2}} \equiv v_A \qquad \text{the 'Alfven Speed'} \tag{5.154}$$

Polarization:

$$E_{\parallel} = E_z = 0, \quad E_x = 0. \quad E_y \neq 0 \quad \Rightarrow V_y = 0 \quad V_x \neq 0 \quad (V_z = 0) \tag{5.155}$$

Party longitudinal (velocity) wave $\rightarrow$ Compression *"Compressional Alfven Wave"*.

2. $N_{\parallel}^2 = \frac{\rho}{\epsilon_0 B^2} = \frac{k_{\parallel}^2 c^2}{\omega^2}$
Any ω has unique $k_{\parallel}$. Wave has unique velocity in $\parallel$ direction: v_A.
Polarization

$$E_z = E_y = 0 \quad E_x \neq 0 \quad \Rightarrow V_x = 0 \quad V_y \neq 0 \quad (V_z = 0) \tag{5.156}$$

Transverse velocity: *"Shear Alfven Wave"*.

Works by field line bending (Tension Force) (no compression).

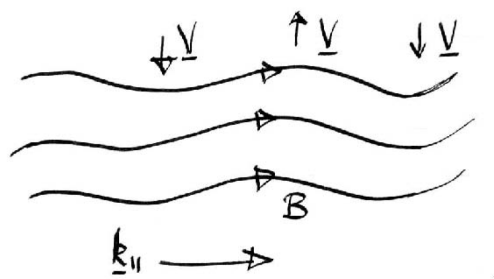

Figure 5.10: Shear Alfven Wave

5.7 Non-Uniform Plasmas and wave propagation

Practical plasmas are not infinite & homogeneous. So how does all this plane wave analysis apply practically?

If the spatial variation of the plasma is slow c.f. the wave length of the wave, then coupling to other waves will be small (negligible).

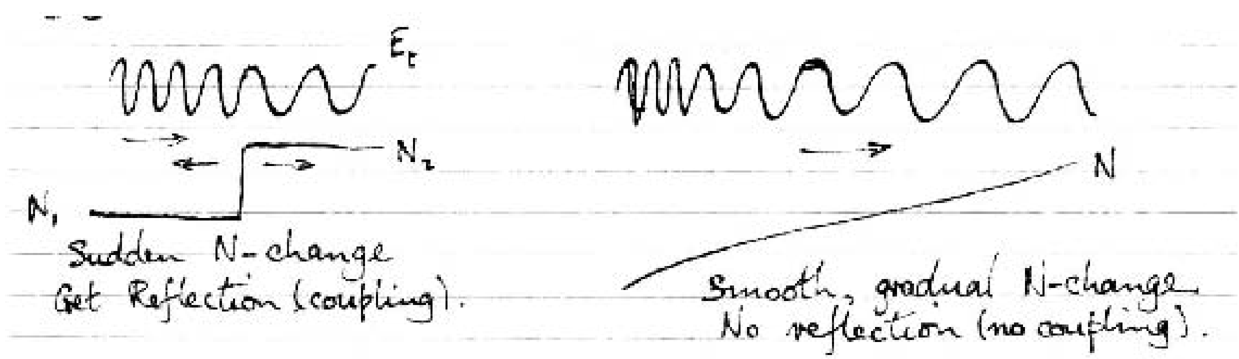

Figure 5.11: Comparison of sudden and gradualy refractive index change.

For a given ω, slowly varying plasma means $N/\frac{dN}{dx} \gg \lambda$ or $kN/\frac{dN}{dx} \gg 1$. Locally, the plasma appears uniform.

Even if the coupling is small, so that locally the wave propagates as if in an infinite uniform plasma, we still need a way of calculating how the solution propagates from one place to the other. This is handled by the 'WKB(J)' or 'eikonal' or 'ray optic' or 'geometric optics' approximation.

WKBJ solution

Consider the model 1-d wave equation (for field ω)

$$\frac{d^2E}{dx^2} + k^2E = 0 \tag{5.157}$$

with k now a slowly varying function of x. Seek a solution in the form

$$E = \exp\left(i\phi\left(x\right)\right) \qquad (-i\omega t \text{ implied}) \tag{5.158}$$

ϕ is the wave phase ($= kx$ in uniform plasma).

Differentiate twice

$$\frac{d^2E}{dx^2} = \{i\frac{d^2\phi}{dx^2} - \left(\frac{d\phi}{dx}\right)^2\}e^{i\phi} \tag{5.159}$$

Substitute into differential equation to obtain

$$\left(\frac{d\phi}{dx}\right)^2 = k^2 + i\frac{d^2\phi}{dx^2} \tag{5.160}$$

Recognize that in uniform plasma $\frac{d^2\phi}{dx^2} = 0$. So in slightly non-uniform, 1st approx is to ignore this term.

$$\frac{d\phi}{dx} \simeq \pm k(x) \tag{5.161}$$

Then obtain a second approximation by substituting

$$\frac{d^2\phi}{dx^2} \simeq \pm\frac{dk}{dx} \tag{5.162}$$

so

$$\left(\frac{d\phi}{dx}\right)^2 \simeq k^2 \pm i\frac{dk}{dx} \tag{5.163}$$

$$\frac{d\phi}{dx} \simeq \pm\left(k \pm \frac{i}{2k}\frac{dk}{dx}\right) \quad \text{using Taylor expansion.} \tag{5.164}$$

Integrate:

$$\phi \simeq \pm\int^x kdx + i\ln\left(k^{\frac{1}{2}}\right) \tag{5.165}$$

Hence E is

$$E = e^{i\phi} = \frac{1}{k^{\frac{1}{2}}}\exp\left(\pm i\int^x kdx\right) \tag{5.166}$$

This is classic WKBJ solution. Originally studied by Green & Liouville (1837), the Green of Green's functions, the Liouville of Sturm Liouville theory.

Basic idea of this approach: (1) solve the local dispersion relation as if in infinite homogeneous plasma, to get $k(x)$, (2) form approximate solution for all space as above.

Phase of wave varies as integral of kdx.

In addition, amplitude varies as $\frac{1}{k^{\frac{1}{2}}}$. This is required to make the total energy flow uniform.

5.8 Two Stream Instability

An example of waves becoming unstable in a non-equilibrium plasma. Analysis is possible using Cold Plasma techniques.

Consider a plasma with two participating cold species but having *different* average velocities.

These are two "streams".

$$\begin{array}{cc} Species1 & Species2 \\ .\rightarrow & . \\ Moving. & Stationary. \\ Speed\ v & \end{array} \tag{5.167}$$

We can look at them in different inertial frames, e.g. species (stream) 2 stationary or 1 stationary (or neither).

We analyse by obtaining the susceptibility for each species and adding together to get total dielectric constant (scalar 1-d if *unmagnetized*).

In a frame of reference in which it is stationary, a stream j has the (Cold Plasma) susceptibility

$$\chi_j = \frac{-\omega_{pj}^2}{\omega^2} \quad . \tag{5.168}$$

If the stream is moving with velocity v_j (*zero order*) then its susceptibility is

$$\chi_j = \frac{-\omega_{pj}^2}{(\omega - kv_j)^2} \quad . \quad (\mathbf{k}\ \&\ \mathbf{v}_j \text{ in same direction}) \tag{5.169}$$

Proof from equation of motion:

$$\frac{q_j}{m_j}\mathbf{E} = \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \mathbf{v}.\nabla\tilde{\mathbf{v}} = (-i\omega + i\mathbf{k}.\mathbf{v}_j)\,\tilde{\mathbf{v}} = -i\,(\omega - kv_j)\,\tilde{\mathbf{v}} \ . \tag{5.170}$$

Current density

$$\mathbf{j} = \rho_j \mathbf{v}_j + \rho_j.\tilde{\mathbf{v}} + \tilde{\rho}\mathbf{v}_j. \tag{5.171}$$

Substitute in

$$\nabla.\mathbf{j} + \frac{\partial \rho}{\partial t} = i\mathbf{k}.\tilde{\mathbf{v}}\rho_j + i\mathbf{k}.\tilde{\mathbf{v}}\rho - i\omega\tilde{\rho} = 0 \tag{5.172}$$

$$\tilde{\rho}_j = \rho_j \frac{\mathbf{k}.\tilde{\mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}_j} \tag{5.173}$$

Hence substituting for $\tilde{\mathbf{v}}$ in terms of $\mathbf{E}$:

$$-\chi_j \epsilon_0 \nabla.\mathbf{E} = \tilde{\rho}_j = \frac{\rho_j q_j}{m_j} \frac{\mathbf{k}.\mathbf{E}}{-i\,(\omega - \mathbf{k}.\mathbf{v}_j)^2}, \tag{5.174}$$

which shows the longitudinal susceptibility is

$$\chi_j = -\frac{\rho_j q_j}{m_j \epsilon_0} \frac{1}{(\omega^2 - kv_j)^2} = \frac{-\omega_{pj}^2}{(\omega - kv_j)^2} \tag{5.175}$$

Proof by transforming frame of reference:

Consider Galileean transformation to a frame moving with the stream at velocity $\mathbf{v}_j$.

$$\mathbf{x} = \mathbf{x}' + \mathbf{v}_j t \quad ; \quad t' = t \tag{5.176}$$

$$\exp i\left(\mathbf{k}.\mathbf{x}-\omega t\right)=\exp i\left(\mathbf{k}.\mathbf{x}'-\left(\omega-\mathbf{k}.\mathbf{v}_j\right)t'\right) \tag{5.177}$$

So in frame of the stream, $\omega'=\omega-\mathbf{k}.\mathbf{v}_j$.
Substitute in stationary cold plasma expression:

$$\chi_j=-\frac{\omega_{pj}^2}{\omega'^2}=-\frac{\omega_{pj}^2}{\left(\omega-kv_j\right)^2}. \tag{5.178}$$

Thus for n streams we have

$$\epsilon=1+\sum_j\chi_j=1-\sum_j\frac{\omega_{pj}^2}{\left(\omega-kv_j\right)^2}\quad. \tag{5.179}$$

Longitudinal wave dispersion relation is

$$\epsilon=0. \tag{5.180}$$

Two streams

$$0=\epsilon=1-\frac{\omega_{p1}^2}{\left(\omega-kv_1\right)^2}-\frac{\omega_{P2}^2}{\left(\omega-kv_2\right)^2} \tag{5.181}$$

For given real k this is a quartic in ω. It has the form:

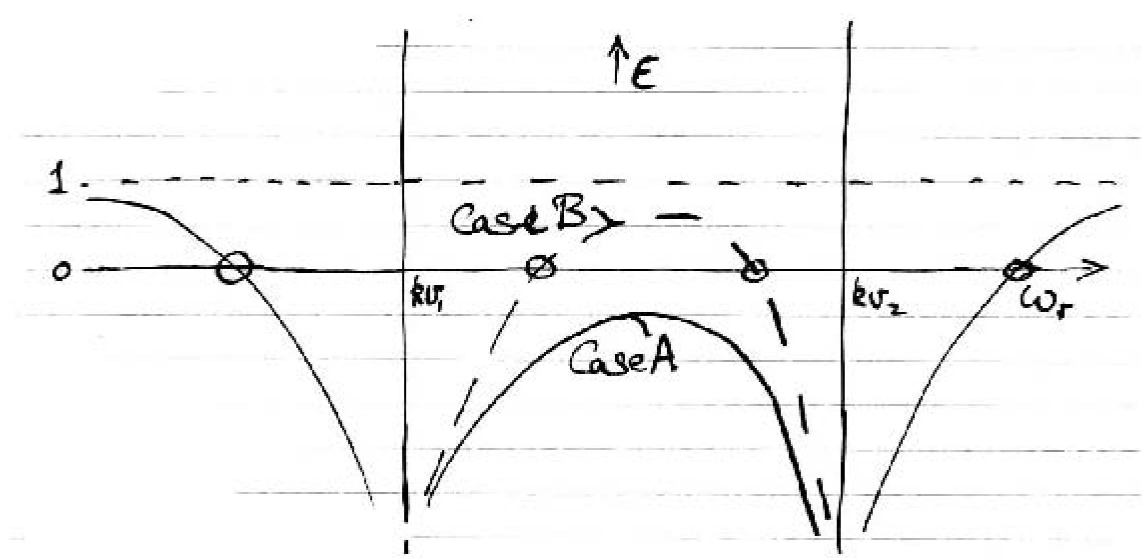

Figure 5.12: Two-stream stability analysis.

If ϵ crosses zero between the wells, then $\exists$ 4 real solutions for ω. (Case B).
If not, then 2 of the solutions are complex: $\omega=\omega_r\pm i\omega_i$ (Case A).
The time dependence of these complex roots is

$$\exp\left(-i\omega t\right)=\exp\left(-i\omega_r t\pm\omega_i t\right). \tag{5.182}$$

The $+ve$ sign is growing in time: *instability*.

It is straightforward to show that Case A occurs if

$$|k(v_2 - v_1)| < \left[\omega_{p1}^{\frac{2}{3}} + \omega_{p2}^{\frac{2}{3}}\right]^{\frac{3}{2}} \quad . \tag{5.183}$$

Small enough k (long enough wavelength) is always unstable.

Simple interpretation ($\omega_{p2}^2 \ll \omega_{p1}^2$, $v_1 = 0$) a tenuous beam in a plasma sees a negative ϵ if $|kv_2| \lesssim \omega_{p1}$.

Negative ϵ implies charge perturbation causes E that enhances itself: charge (spontaneous) bunching.

5.9 Kinetic Theory of Plasma Waves

Wave damping is due to wave-particle resonance. To treat this we need to keep track of the particle distribution in velocity space → kinetic theory.

5.9.1 Vlasov Equation

Treat particles as moving in 6-D phase space $\mathbf{x}$ position, $\mathbf{v}$ velocity. At any instant a particle occupies a unique position in phase space $(\mathbf{x}, \mathbf{v})$.

Consider an elemental volume $d^3\mathbf{x}d^3\mathbf{v}$ of phase space $[dxdydzdv_xdv_ydv_z]$, at $(\mathbf{x}, \mathbf{v})$. Write down an equation that is conservation of particles for this volume

$$\begin{aligned} -\frac{\partial}{\partial t}\left(fd^3\mathbf{x}d^3\mathbf{v}\right) &= \left[v_x f\left(\mathbf{x} + dx\hat{\mathbf{x}}, \mathbf{v}\right) - v_x f\left(\mathbf{x}, \mathbf{v}\right)\right] dydzd^3\mathbf{v} \\ &+ \text{ same for } dy, dz \\ &+ \left[a_x f\left(\mathbf{x}, \mathbf{v} + dv_x\hat{\mathbf{x}}\right) - a_x f\left(\mathbf{x}, \mathbf{v}\right)\right] d^3\mathbf{x}dv_ydv_z \\ &+ \text{ same for } dv_y, dv_z \end{aligned} \tag{5.184}$$

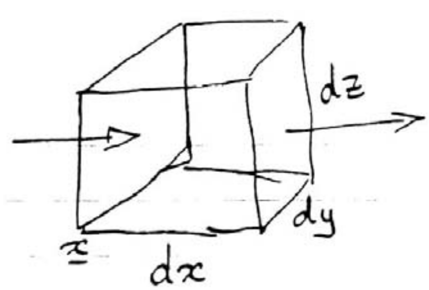

Figure 5.13: Difference in flow across x-surfaces $(+y + z)$.

$\mathbf{a}$ is "velocity space motion", i.e. acceleration.

Divide through by $d^3\mathbf{x}d^3\mathbf{v}$ and take limit

$$\begin{aligned} -\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}(v_x f) + \frac{\partial}{\partial y}(v_y f) + \frac{\partial}{\partial z}(v_z f) + \frac{\partial}{\partial v_x}(a_x f) + \frac{\partial}{\partial v_y}(a_y f) + \frac{\partial}{\partial v_z}(a_z f) \\ &= \nabla.(\mathbf{v}f) + \nabla_v.(\mathbf{a}f) \end{aligned} \tag{5.185}$$

[Notation: Use $\frac{\partial}{\partial \mathbf{x}} \leftrightarrow \nabla$; $\frac{\partial}{\partial \mathbf{v}} \leftrightarrow \nabla_v$].
Take this simple continuity equation in phase space and expand:

$$\frac{\partial f}{\partial t} + (\nabla.\mathbf{v})\, f + (\mathbf{v}.\nabla)\, f + (\nabla_v.\mathbf{a})\, f + (\mathbf{a}.\nabla_v)\, f = 0. \tag{5.186}$$

Recognize that ∇ means here $\frac{\partial}{\partial x}$ etc. *keeping* $\mathbf{v}$ *constant* so that $\nabla.\mathbf{v} = 0$ by definition. So

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial f}{\partial \mathbf{v}} = -f\,(\nabla_v.\mathbf{a}) \tag{5.187}$$

Now we want to couple this equation with Maxwell's equations for the fields, and the Lorentz force

$$\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) \tag{5.188}$$

Actually we don't want to use the **E** retaining all the local effects of individual particles. We want a smoothed out field. Ensemble averaged **E**.
Evaluate

$$\begin{aligned} \nabla_v.\mathbf{a} &= \nabla_v.\frac{q}{m}(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) = \frac{q}{m}\nabla_v.(\mathbf{v}\wedge\mathbf{B}) &(5.189) \\ &= \frac{q}{m}\mathbf{B}.(\nabla_v\wedge\mathbf{v}) = 0. &(5.190) \end{aligned}$$

So RHS is zero. However in the use of smoothed out E we have ignored local effect of one particle on another due to the graininess. That is *collisions*.

Boltzmann Equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \tag{5.191}$$

Vlasov Equation ≡ Boltzman Eq *without* collisions. For electromagnetic forces:

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E} + \mathbf{v}\wedge\mathbf{B})\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{5.192}$$

Interpretation:
Distribution function is constant along particle orbit in phase space: $\frac{d}{dt}f = 0$.

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt}.\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{dt}.\frac{\partial f}{\partial \mathbf{v}} \tag{5.193}$$

Coupled to Vlasov equation for each particle species we have Maxwell's equations.

Vlasov-Maxwell Equations

$$\frac{\partial f_j}{\partial t} + \mathbf{v}.\frac{\partial f_j}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right).\frac{\partial f_j}{\partial \mathbf{v}_j} = 0 \tag{5.194}$$

$$\nabla\wedge\mathbf{E} = \frac{-\partial\mathbf{B}}{\partial t} \quad , \quad \nabla\wedge\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \tag{5.195}$$

$$\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla.\mathbf{B} = 0 \tag{5.196}$$

Coupling is completed via charge & current densities.

$$\rho = \sum_j q_j n_j = \sum_J q_j \int f_j d^3\mathbf{v} \tag{5.197}$$

$$\mathbf{j} = \sum_j q_j n_j \mathbf{V}_j = \sum_j q_j \int f_j \mathbf{v} d^3\mathbf{v}. \tag{5.198}$$

Describe phenomena in which collisions are not important, keeping track of the (statistically averaged) particle distribution function.

Plasma waves are the most important phenomena covered by the Vlasov-Maxwell equations.

6-dimensional, nonlinear, time-dependent, integral-differential equations!

5.9.2 Linearized Wave Solution of Vlasov Equation

Unmagnetized Plasma

Linearize the Vlasov Eq by supposing

$$f = f_0(\mathbf{v}) + f_1(v)\exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \quad , \quad f_1 \text{ small.} \tag{5.199}$$

$$\text{also } \mathbf{E} = \mathbf{E}_1 \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \quad \mathbf{B} = \mathbf{B}_1 \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right) \tag{5.200}$$

Zeroth order f_0 equation satisfied by $\frac{\partial}{\partial t}, \frac{\partial}{\partial x} = 0$. First order:

$$-i\omega f_1 + \mathbf{v}.i\mathbf{k}f_1 + \frac{q}{m}\left(\mathbf{E}_1 + \mathbf{v}\wedge\mathbf{B}_1\right).\frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{5.201}$$

[Note $\mathbf{v}$ is not per se of any order, it is an independent variable.]

Solution:

$$f_1 = \frac{1}{i\left(\omega - \mathbf{k}.\mathbf{v}\right)}\frac{q}{m}\left(\mathbf{E}_1 + \mathbf{v}\wedge\mathbf{B}_1\right).\frac{\partial f_0}{\partial \mathbf{v}} \tag{5.202}$$

For convenience, assume f_0 is *isotropic*. Then $\frac{\partial f_0}{\partial \mathbf{v}}$ is in direction $\mathbf{v}$ so $\mathbf{v}\wedge\mathbf{B}_1.\frac{\partial f_0}{\partial \mathbf{v}} = 0$

$$f_1 = \frac{\frac{q}{m}\mathbf{E}_1.\frac{\partial f_0}{\partial \mathbf{v}}}{i\left(\omega - \mathbf{k}.\mathbf{v}\right)} \tag{5.203}$$

We want to calculate the conductivity $\boldsymbol{\sigma}$. Do this by simply integrating:

$$\mathbf{j} = \int q f_1 \mathbf{v} d^3 v = \frac{q^2}{im}\int \frac{\mathbf{v}\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}} d^3 v \ .\mathbf{E}_1. \tag{5.204}$$

Here the electric field has been taken outside the v-integral but its dot product is with $\partial f_0/\partial\mathbf{v}$. Hence we have the tensor conductivity,

$$\boldsymbol{\sigma} = \frac{q^2}{im}\int \frac{\mathbf{v}\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}.\mathbf{v}} d^3v \tag{5.205}$$

Focus on zz component:

$$1 + \chi_{zz} = \epsilon_{zz} = 1 + \frac{\sigma_{zz}}{-i\omega\epsilon_0} = 1 + \frac{q^2}{\omega m \epsilon_0}\int \frac{v_z\frac{\partial f_0}{\partial v_z}}{\omega - \mathbf{k}.\mathbf{v}} d^3\mathbf{v} \tag{5.206}$$

Such an expression applies for the conductivity (susceptibility) of each species, if more than one needs to be considered.

It looks as if we are there! Just do the integral!

Now the problem becomes evident. The integrand has a zero in the denominator. At least we can do 2 of 3 integrals by defining the 1-dimensional distribution function

$$f_z(v_z) \equiv \int f(\mathbf{v}) dv_x dv_y \qquad (\mathbf{k} = k\hat{\mathbf{z}}) \tag{5.207}$$

Then

$$\chi = \frac{q^2}{\omega m \epsilon_0}\int \frac{v_z\frac{\partial f_z}{\partial v_z}}{\omega - kv_z} dv_z \tag{5.208}$$

(drop the z suffix from now on. 1-d problem).

How do we integrate through the pole at $v = \frac{\omega}{k}$? Contribution of resonant particles. Crucial to get right.

Path of velocity integration

First, realize that the solution we have found is not complete. In fact a more general solution can be constructed by adding any solution of

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z} = 0 \tag{5.209}$$

[We are dealing with 1-d Vlasov equation: $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0$.] Solution of this is

$$f_1 = g(vt - z, v) \tag{5.210}$$

where g is an arbitrary function of its arguments. Hence general solution is

$$f_1 = \frac{\frac{q}{m}E\frac{\partial f_0}{\partial v}}{i\left(\omega - kv\right)} \exp i\left(kz - \omega t\right) + g\left(vt - z, v\right) \tag{5.211}$$

and g must be determined by initial conditions. In general, if we start up the wave suddenly there will be a transient that makes g non-zero.

So instead we consider a case of *complex* ω (real k for simplicity) where $\omega = \omega_r + i\omega_i$ and $\omega_i > 0$.

This case corresponds to a growing wave:

$$\exp(-i\omega t) = \exp(-i\omega_r t + \omega_i t) \tag{5.212}$$

Then we can take our initial condition to be $f_1 = 0$ at $t \to -\infty$. This is satisfied by taking $g = 0$.

For $\omega_i > 0$ the complementary function, g, is zero.

Physically this can be thought of as treating a case where there is a very gradual, smooth start up, so that no transients are generated.

Thus if $\omega_i > 0$, the solution is simply the velocity integral, taken along the real axis, with no additional terms. For

$$\omega_i > 0, \qquad \chi = \frac{q^2}{\omega m \epsilon_o} \int_C \frac{v \frac{\partial f}{\partial v}}{\omega - kv} dv \tag{5.213}$$

where there is now no difficulty about the integration because ω is complex.

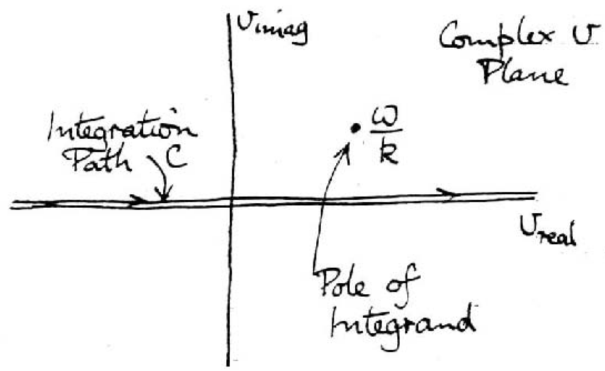

Figure 5.14: Contour of integration in complex v-plane.

The pole of the integrand is at $v = \frac{\omega}{k}$ which is above the real axis.

The question then arises as to how to do the calculation if $\omega_i \leq 0$. The answer is by "analytic continuation", regarding all quantities as complex.

"Analytic Continuation" of χ is accomplished by allowing ω/k to move (e.g. changing the ω_i) but never allowing any poles to cross the integration contour, as things change continuously.

Remember (Fig 5.15)

$$\oint_c F dz = \sum \text{residues} \ \times 2\pi i \tag{5.214}$$

(Cauchy's theorem)

Where residues $= \lim_{z \to z_k}[F(z)/(z - z_k)]$ at the poles, z_k, of $F(z)$. We can deform the contour how we like, provided no poles cross it. Hence contour (Fig 5.16)

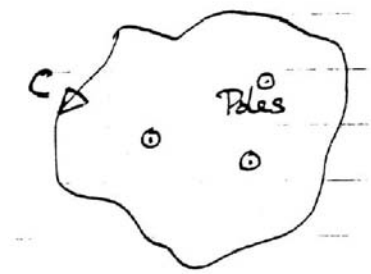

Figure 5.15: Cauchy's theorem.

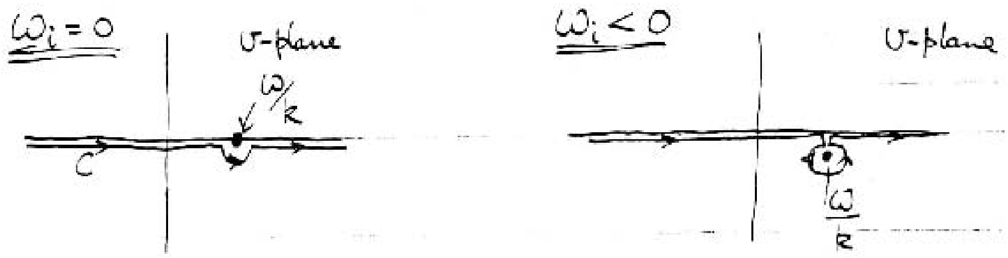

Figure 5.16: Landau Contour

We conclude that the integration contour for $\omega_i < 0$ is *not* just along the real v axis. It includes the pole also.

To express our answer in a universal way we use the notation of "Principal Value" of a singular integral defined as the average of paths above and below

$$\wp \int \frac{F}{v - v_0} dv = \frac{1}{2} \left[\int_{C_1} + \int_{C_2} \right] \frac{F}{v - v_0} dv \tag{5.215}$$

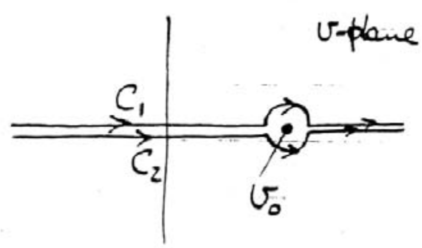

Figure 5.17: Two halves of principal value contour.

Then

$$\chi = \frac{1^2}{\omega m \epsilon_o} \{ \wp \int \frac{v \frac{\partial f_0}{\partial v}}{\omega - kv} dv - \frac{1}{2} 2\pi i \frac{\omega}{k^2} \left. \frac{\partial f_0}{\partial v} \right|_{v = \frac{\omega}{k}} \} \tag{5.216}$$

Second term is half the normal residue term; so it is half of the integral round the pole.

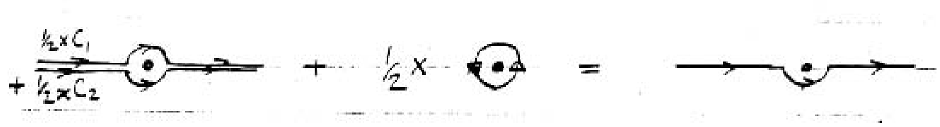

Figure 5.18: Contour equivalence.

Our expression is only short-hand for the (Landau) prescription:
"Integrate below the pole". (Nautilus).

Contribution from the pole can be considered to arise from the complementary function $g(vt-z,v)$. If g is to be proportional to $\exp(ikz)$, then it must be of the form $g = \exp[ik(z-vt)]h(v)$ where $h(v)$ is an arbitrary function. To get the result previously calculated, the value of $h(v)$ must be (for real ω)

$$h(v) = \pi \frac{q}{m}\frac{1}{k}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega}{k}} \delta\left(v - \frac{\omega}{k}\right) \tag{5.217}$$

$$\left(\text{so that } \int \frac{q}{-i\omega\epsilon_o} vg dv = \left(\pi i \frac{\omega}{k^2}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega}{k}}\right)\frac{q^2}{\omega m \epsilon_o} \quad .\right) \tag{5.218}$$

This Dirac delta function says that the complementary function is limited to particles with "exactly" the wave phase speed $\frac{\omega}{k}$. It is the resonant behaviour of these particles and the imaginary term they contribute to χ that is responsible for wave damping.

We shall see in a moment, that the standard case will be $\omega_i < 0$, so the opposite of the prescription $\omega_i > 0$ that makes $g = 0$. Therefore there will generally be a complementary function, non-zero, describing resonant effects. We don't have to calculate it explicitly because the Landau prescription takes care of it.

5.9.3 Landau's original approach. (1946)

Corrected Vlasov's assumption that the correct result was just the principal value of the integral. Landau recognized the importance of initial conditions and so used Laplace Transform approach to the problem

$$\tilde{A}(p) = \int_0^\infty e^{-pt} A(t) dt \tag{5.219}$$

The Laplace Transform inversion formula is

$$A(t) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} e^{pt}\tilde{A}(p) dp \tag{5.220}$$

where the path of integration must be chosen to the right of any poles of $\tilde{A}(p)$ (i.e. s large enough). Such a prescription seems reasonable. If we make $\Re(p)$ large enough then the $\tilde{A}(p)$ integral will presumably exist. The inversion formula can also be proved rigorously so that gives confidence that this is the right approach.

If we identify $p \to -i\omega$, then the transform is $\tilde{A} = \int e^{i\omega t} A(t) dt$, which can be identified as the Fourier transform that would give component $\tilde{A} \propto e^{-i\omega t}$, the wave we are discussing. Making $\Re(p)$ positive enough to be to the right of all poles is then equivalent to making $\Im(\omega)$ positive enough so that the path in ω-space is above all poles, in particular $\omega_i > \Im(kv)$. For real velocity, v, this is precisely the condition $\omega_i > 0$, we adopted before to justify putting the complementary function zero.

Either approach gives the same prescription. It is all bound up with satisfying causality.

5.9.4 Solution of Dispersion Relation

We have the dielectric tensor

$$\epsilon = 1 + \chi = 1 + \frac{q^2}{\omega m \epsilon_0} \left\{ \wp \int \frac{v \frac{\partial f_0}{\partial v}}{\omega - kv} dv - \pi i \frac{\omega}{k^2} \left. \frac{\partial f_0}{\partial v} \right|_{\frac{\omega}{k}} \right\} \quad , \tag{5.221}$$

for a general isotropic distribution. We also know that the dispersion relation is

$$\begin{bmatrix} -N^2 + \epsilon_t & 0 & 0 \\ 0 & -N^2 + \epsilon_t & 0 \\ 0 & 0 & \epsilon \end{bmatrix} = \left(-N^2 + \epsilon_t \right)^2 \epsilon = 0 \tag{5.222}$$

Giving transverse waves $N^2 = \epsilon_t$ and
longitudinal waves $\epsilon = 0$.
Need to do the integral and hence get ϵ.

Presumably, if we have done this right, we ought to be able to get back the cold-plasma result as an approximation in the appropriate limits, plus some corrections. We previously argued that cold-plasma is valid if $\frac{\omega}{k} \gg v_t$. So regard $\frac{kv}{\omega}$ as a small quantity and expand:

$$\begin{aligned} \wp \int \frac{v \frac{\partial f_0}{dv}}{\omega \left(1 - \frac{kv}{\omega} \right)} dv &= \frac{1}{\omega} \int v \frac{\partial f_0}{\partial v} \left[1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega} \right)^2 + ... \right] dv \\ &= \frac{-1}{\omega} \int f_o \left[1 + \frac{2kv}{\omega} + 3 \left(\frac{kv}{\omega} \right)^2 + ... \right] dv \quad \text{(by parts)} \\ &\simeq \frac{-1}{\omega} \left[n + \frac{3nT}{m} \frac{k^2}{\omega^2} \right] + ... \end{aligned} \tag{5.223}$$

Here we have assumed we are in the particles' average rest frame (no bulk velocity) so that $\int f_0 v dv = 0$ and also we have used the temperature definition

$$nT = \int m v^2 f_0 dv \quad , \tag{5.224}$$

appropriate to one degree of freedom (1-d problem). Ignoring the higher order terms we get:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} \left\{ 1 + 3 \frac{T}{m} \frac{k^2}{\omega^2} + \pi i \frac{\omega^2}{k^2} \frac{1}{n} \left. \frac{\partial f_0}{\partial v} \right|_{\frac{\omega}{k}} \right\} \tag{5.225}$$

This is just what we expected. Cold plasma value was $\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$. We have two corrections

1. To real part of ϵ, correction $3\frac{T}{m}\frac{k^2}{\omega^2} = 3\left(\frac{v_t}{v_p}\right)^2$ due to finite temperature. We could have got this from a fluid treatment *with* pressure.

2. Imaginary part $\rightarrow$ antihermitian part of $\epsilon \rightarrow$ dissipation.

Solve the dispersion relation for longitudinal waves $\epsilon = 0$ (again assuming k real ω complex). Assume $\omega_i \ll \omega_r$ then

$$\begin{aligned}
(\omega_r + i\omega_i)^2 &\simeq \omega_r^2 + 2\omega_r\omega_i i = \omega_p^2\{1 + 3\frac{T}{m}\frac{k^2}{\omega^2} + \pi i \frac{\omega^2}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}}\} \\
&\simeq \omega_p^2\{1 + 3\frac{T}{m}\frac{k^2}{\omega_r^2} + \pi i \frac{\omega_r^2}{k^2}\frac{1}{n}\frac{\partial f_o}{\partial v}|_{\frac{\omega_r}{k}}\} \qquad (5.226) \\
\text{Hence } \omega_1 &\simeq \frac{1}{2\omega_r i}\omega_p^2 \pi i \frac{\omega_r^2}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega_r}{k}} = \omega_p^2\frac{\pi}{2}\frac{\omega_r}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega_r}{k}} \qquad (5.227)
\end{aligned}$$

For a Maxwellian distribution

$$f_0 = \left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \exp\left(-\frac{mv^2}{2T}\right) n \qquad (5.228)$$

$$\frac{\partial f_0}{\partial v} = \left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \left(-\frac{mv}{T}\right) \exp\left(-\frac{mv^2}{2T}\right) n \qquad (5.229)$$

$$\omega_i \simeq -\omega_p^2 \frac{\pi}{2}\frac{\omega_r^2}{k^3}\left(\frac{m}{2\pi T}\right)^{\frac{1}{2}} \frac{m}{T} \exp\left(-\frac{m\omega_r^2}{2Tk^2}\right) \qquad (5.230)$$

The difference between ω_r and ω_p may not be important in the outside but ought to be retained inside the exponential since

$$\frac{m}{2T}\frac{\omega_p^2}{k^2}\left[1 + 3\frac{T}{m}\frac{k^2}{\omega_p^2}\right] = \frac{m\omega_p^2}{2Tk^2} + \frac{3}{2} \qquad (5.231)$$

$$\text{So} \quad \omega_i \simeq -\omega_p \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \frac{\omega_p^3}{k^3}\frac{1}{v_t^3} \exp\left(-\frac{m\omega_p^2}{2Tk^2} - \frac{3}{2}\right) \qquad (5.232)$$

Imaginary part of ω is *negative* $\Rightarrow$ damping. This is Landau Damping.

Note that we have been treating a single species (electrons by implication) but if we need more than one we simply add to χ. Solution is then more complex.

5.9.5 Direct Calculation of Collisionless Particle Heating

(Landau Damping without complex variables!)

We show by a direct calculation that net energy is transferred to electrons.

Suppose there exists a longitudinal wave

$$\mathbf{E} = E\cos(kz - \omega t)\hat{\mathbf{z}} \tag{5.233}$$

Equations of motion of a particle

$$\frac{dv}{dt} = \frac{q}{m}E\cos(kz - \omega t) \tag{5.234}$$

$$\frac{dz}{dt} = v \tag{5.235}$$

Solve these assuming E is small by a perturbation expansion $v = v_0 + v_1 + ...$, $z = z_0(t) + z_1(t) + ...$.

Zeroth order:

$$\frac{dv_o}{dt} = 0 \Rightarrow v_0 = \text{const} \quad , \quad z_0 = z_i + v_0 t \tag{5.236}$$

where $z_i = \text{const}$ is the initial position.

First Order

$$\frac{dv_1}{dt} = \frac{q}{m}E\cos\left(kz_0 - \omega t\right) = \frac{q}{m}E\cos\left(k\left(z_i + v_0 t\right) - \omega t\right) \tag{5.237}$$

$$\frac{dz_1}{dt} = v_1 \tag{5.238}$$

Integrate:

$$v_1 = \frac{qE}{m}\frac{\sin\left(kz_i + kv_0 - \omega t\right)}{kv_0 - \omega} + \text{const.} \tag{5.239}$$

take initial conditions to be $v_1, v_2 = 0$. Then

$$v_1 = \frac{qE}{m}\frac{\sin\left(kz_i + \Delta\omega t\right) - \sin\left(kz_i\right)}{\Delta\omega} \tag{5.240}$$

where $\Delta\omega \equiv kv_0 - \omega$, is (-) the frequency at which the particle feels the wave field.

$$z_1 = \frac{qE}{m}\left[\frac{\cos kz_i - \cos\left(kz_i + \Delta\omega t\right)}{\Delta\omega^2} - t\frac{\sin kz_i}{\Delta\omega}\right] \tag{5.241}$$

(using $z_1(0) = 0$).

2nd Order (Needed to get energy right)

$$\begin{aligned}\frac{dv_2}{dt} &= \frac{qE}{M}\{\cos\left(kz_i + kv_0 t - \omega t + kz_1\right) - \cos\left(kz_i + kv_0 t - \omega t\right)\} \\ &= \frac{qE}{m}kz_i\{-\sin\left(kz_i + \Delta\omega t\right)\} \qquad (kz_1 \ll 1)\end{aligned} \tag{5.242}$$

Now the gain in kinetic energy of the particle is

$$\begin{aligned}\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 &= \frac{1}{2}m\{(v_0 + v_1 + v_2 + ...)^2 - v_0^2\} \\ &= \frac{1}{2}\{2v_0 v_1 + v_1^2 + 2v_0 v_2 + \text{higher order}\}\end{aligned} \tag{5.243}$$

and the rate of increase of K.E. is

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\left(v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt}\right) \tag{5.244}$$

We need to average this over space, i.e. over z_i. This will cancel any component that simply oscillates with z_i.

$$\left\langle\frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle = \left\langle v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt}\right\rangle m \tag{5.245}$$

$$\begin{aligned}
\left\langle v_0\frac{dv_1}{dt}\right\rangle &= 0 && (5.246)\\
\left\langle v_1\frac{dv_1}{dt}\right\rangle &= \left\langle\frac{q^2E^2}{m^2}\left[\frac{\sin(kz_i+\Delta\omega t)-\sin kz_i}{\Delta\omega}\cos(kz_i+\Delta\omega t)\right]\right\rangle \\
&= \frac{q^2E^2}{m^2}\left\langle\frac{\sin(kz_i+\Delta\omega t)-\sin(kz_i+\Delta\omega t)\cos\Delta\omega t+\cos(kz_i+\Delta\omega t)\sin\Delta\omega t}{\Delta\omega}\right. \\
&\qquad \left.\cos(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\left\langle\frac{\sin\Delta\omega t}{\Delta\omega}\cos^2(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\frac{1}{2}\frac{\sin\Delta\omega t}{\Delta\omega} && (5.247)\\
\left\langle v_0\frac{dv_2}{dt}\right\rangle &= \frac{-q^2E^2}{m^2}kv_0\left\langle\left(\frac{\cos kz_i-\cos(kz_i+\Delta\omega t)}{\Delta\omega^2}-t\frac{\sin kz_i}{\Delta\omega}\right)\sin(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{-q^2E^2}{m^2}kv_0\left\langle\left(\frac{\sin\Delta\omega t}{\Delta\omega^2}-t\frac{\cos\Delta\omega t}{\Delta\omega}\right)\sin^2(kz_i+\Delta\omega t)\right\rangle \\
&= \frac{q^2E^2}{m^2}\frac{kv_0}{2}\left[-\frac{\sin\Delta\omega t}{\Delta\omega^2}+t\frac{\cos\Delta\omega t}{\Delta\omega}\right] && (5.248)
\end{aligned}$$

Hence

$$\begin{aligned}
\left\langle\frac{d}{dt}\frac{1}{2}mv^2\right\rangle &= \frac{q^2E^2}{2m}\left[\frac{\sin\Delta\omega t}{\Delta\omega}-kv_0\frac{\sin\Delta\omega t}{\Delta\omega^2}+kv_0t\frac{\cos\Delta\omega t}{\Delta\omega}\right] && (5.249)\\
&= \frac{q^2E^2}{2m}\left[\frac{-\omega\sin\Delta\omega t}{\Delta\omega^2}+\frac{\omega t}{\Delta\omega}\cos\Delta\omega t+t\cos\Delta\omega t\right] && (5.250)
\end{aligned}$$

This is the space-averaged power into particles of a specific velocity v_0. We need to integrate over the distribution function. A trick identify helps:

$$\begin{aligned}
\frac{-\omega}{\Delta\omega^2}\sin\Delta\omega t \quad + \quad \frac{\omega t}{\Delta\omega}\cos\Delta\omega t+t\cos\Delta\omega t &= \frac{\partial}{\partial\Delta\omega}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega}+\sin\Delta\omega t\right) && (5.251)\\
&= \frac{1}{k}\frac{\partial}{\partial v_0}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega}+\sin\Delta\omega t\right) && (5.252)
\end{aligned}$$

Hence power per unit volume is

$$\begin{aligned} P &= \int \left\langle \frac{d}{dt}\frac{1}{2}mv^2 \right\rangle f(v_0)\, dv_0 \\ &= \frac{q^2E^2}{2mk}\int f(v_0)\frac{\partial}{\partial v_0}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right) dv_0 \\ &= -\frac{q^2E^2}{2mk}\int \left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right)\frac{\partial f}{\partial v_0} dv_0 \end{aligned} \tag{5.253}$$

As t becomes large, $\sin\Delta\omega t = \sin(kv_0 - \omega)t$ becomes a rapidly oscillating function of v_0. Hence second term of integrand contributes negligibly and the first term,

$$\propto \frac{\omega\sin\Delta\omega t}{\Delta\omega} = \frac{\sin\Delta\omega t}{\Delta\omega t}\quad \omega t \tag{5.254}$$

becomes a highly localized, delta-function-like quantity. That enables the rest of the integrand to be evaluated just where $\Delta\omega = 0$ (i.e. $kv_0 - \omega = 0$).

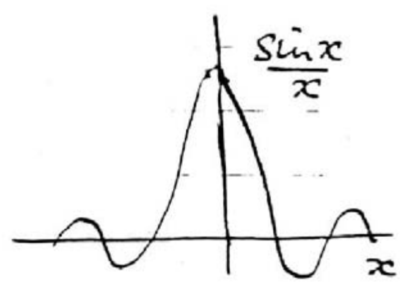

Figure 5.19: Localized integrand function.

So:

$$P = -\frac{q^2E^2}{2mk}\frac{\omega}{k}\frac{\partial f}{\partial v}|_{\frac{\omega}{k}}\int\frac{\sin x}{x}dx \tag{5.255}$$

$x = \Delta\omega t = (kv_0 - \omega)t$.
and $\int \frac{\sin x}{x} dz = \pi$ so

$$P = -E\frac{\pi q^2\omega}{2mk^2}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}} \tag{5.256}$$

We have shown that there is a net transfer of energy to particles at the resonant velocity $\frac{\omega}{k}$ from the wave. (Positive if $\frac{\partial f}{\partial v}|$ is negative.)

5.9.6 Physical Picture

$\Delta\omega$ is the frequency in the particles' (unperturbed) frame of reference, or equivalently it is kv_0' where v_0' is particle speed in wave frame of reference. The latter is easier to deal with. $\Delta\omega t = kv_0't$ is the phase the particle travels in time t. We found that the energy gain was of the form

$$\int\frac{\sin\Delta\omega t}{\Delta\omega t}d(\Delta\omega t)\,. \tag{5.257}$$

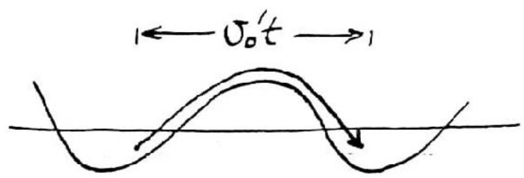

Figure 5.20: Phase distance traveled in time t.

This integrand becomes small (and oscillatory) for $\Delta\omega t \gg 1$. Physically, this means that if particle moves through many wavelengths its energy gain is small. Dominant contribution is from $\Delta\omega t < \pi$. These are particles that move through less than $\frac{1}{2}$ wavelength during the period under consideration. These are the resonant particles.

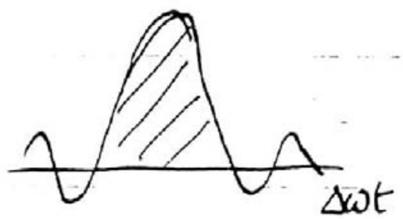

Figure 5.21: Dominant contribution

Particles moving slightly *faster* than wave are *slowed* down. This is a second-order effect.

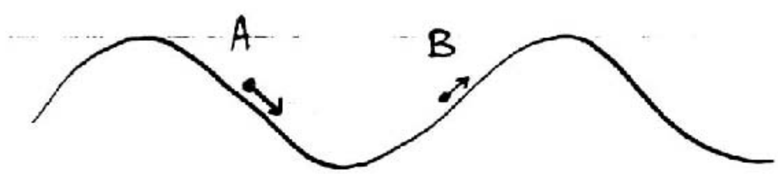

Figure 5.22: Particles moving slightly faster than the wave.

Some particles of this v_0 group are being accelerated (A) some slowed (B). Because A's are then going faster, they spend less time in the 'down' region. B's are slowed; they spend more time in up region. Net effect: tendency for particle to move its speed toward that of wave.

Particles moving slightly *slower* than wave are *speeded* up. (Same argument). But this is only true for particles that have "caught the wave".

Summary: Resonant particles' velocity is drawn toward the wave phase velocity.

Is there net energy when we average both slower and faster particles? Depends which type has most.

Our Complex variables wave treatment and our direct particle energy calculation give consistent answers. To show this we need to show energy conservation. Energy density of

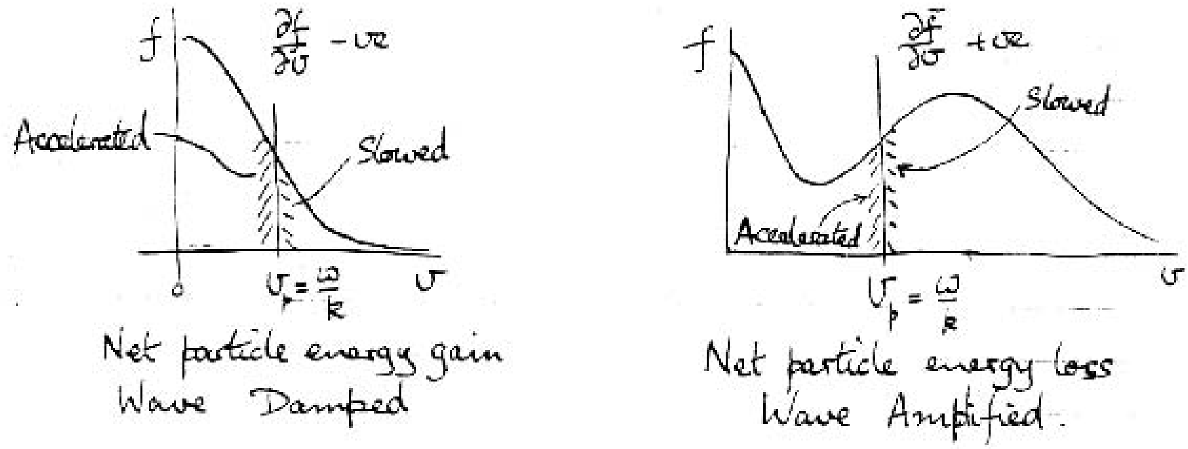

Figure 5.23: Damping or growth depends on distribution slope

wave:

$$\overline{W} = \underbrace{\frac{1}{2}}_{<\sin^2>} [\ \underbrace{\frac{1}{2}\epsilon_0|E^2|}_{Electrostatic} \ + \ \underbrace{n\frac{1}{2}m|\tilde{v}^2|}_{Particle\ Kinetic}\ ] \tag{5.258}$$

Magnetic wave energy zero (negligible) for a longitudinal wave. We showed in Cold Plasma treatment that the velocity due to the wave is $\tilde{v} = \frac{qE}{-i\omega m}$ Hence

$$\overline{W} \simeq \frac{1}{2}\frac{\epsilon_0 E^2}{2}\left[1 + \frac{\omega_p^2}{\omega^2}\right] \qquad \text{(again electrons only)} \tag{5.259}$$

When the wave is damped, it has imaginary part of ω, ω_i and

$$\frac{d\overline{W}}{dt} = \overline{W}\frac{1}{E^2}\frac{dE^2}{dt} = 2\omega_i\overline{W} \tag{5.260}$$

Conservation of energy requires that this equal minus the particle energy gain rate, P. Hence

$$\omega_i = \frac{-P}{2W} = \frac{+E^2\frac{\pi q^2\omega}{2mk^2}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}}}{\frac{\epsilon_0 E^2}{1}\left[1 + \frac{\omega_p^2}{\omega^2}\right]} = \omega_p^2\frac{\pi}{2}\frac{\omega}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\frac{\omega}{k}} \times \frac{2}{1 + \frac{\omega_p^2}{\omega^2}} \tag{5.261}$$

So for waves such that $\omega \sim \omega_p$, which is the dispersion relation to lowest order, we get

$$\omega_i = \omega_p^2\frac{\pi}{2}\frac{\omega_r}{k^2}\frac{1}{n}\left.\frac{\partial f_0}{\partial v}\right|_{\frac{\omega_r}{k}} \quad . \tag{5.262}$$

This exactly agrees with the damping calculated from the complex dispersion relation using the Vlasov equation.

This is the Landau damping calculation for longitudinal waves in a (magnetic) field-free plasma. Strictly, just for electron plasma waves.

How does this apply to the general magnetized plasma case with multiple species?

Doing a complete evaluation of the dielectric tensor using kinetic theory is feasible but very heavy algebra. Our direct intuitive calculation gives the correct answer more directly.

5.9.7 Damping Mechanisms

Cold plasma dielectric tensor is Hermitian. [Complex conjugate*, transpose^T = original matrix.] This means *no damping* (dissipation).

The proof of this fact is simple but instructive. Rate of doing work on plasma per unit volume is $P = \mathbf{E}.\mathbf{j}$. However we need to observe notation.

Notation is that $\mathbf{E}(\mathbf{k}, \omega)$ is amplitude of wave which is really $\Re(\mathbf{E}(\mathbf{k}, \omega) \exp i(\mathbf{k}.\mathbf{x} - \omega t))$ and similarly for $\mathbf{j}$. Whenever products are taken: must *take real part first.* So

$$\begin{aligned} P &= \Re\left(\mathbf{E} \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right)\right).\Re\left(\mathbf{j} \exp i\left(\mathbf{k}.\mathbf{x} - \omega t\right)\right) \\ &= \frac{1}{2}\left[\mathbf{E}e^{i\phi} + \mathbf{E}^* e^{-i\phi}\right].\frac{1}{2}\left[\mathbf{j}e^{i\phi} + \mathbf{j}^* e^{-i\phi}\right] \quad (\phi = \mathbf{k}.\mathbf{x} - \omega t.) \\ &= \frac{1}{4}\left[\mathbf{E}.\mathbf{j}e^{2i\phi} + \mathbf{E}.\mathbf{j}^* + \mathbf{E}^*.\mathbf{j} + \mathbf{E}^*.\mathbf{j}^* e^{-2i\phi}\right] \end{aligned} \tag{5.263}$$

The terms $e^{2i\phi}$ & $e^{-2i\phi}$ are rapidly varying. We usually average over at least a period. These average to zero. Hence

$$\langle P \rangle = \frac{1}{4}\left[\mathbf{E}.\mathbf{j}^* + \mathbf{E}^*.\mathbf{j}\right] = \frac{1}{2}\Re\left(\mathbf{E}.\mathbf{j}^*\right) \tag{5.264}$$

Now recognize that $\mathbf{j} = \boldsymbol{\sigma}.\mathbf{E}$ and substitute

$$\langle P \rangle = \frac{1}{4}\left[\mathbf{E}.\boldsymbol{\sigma}^*.\mathbf{E}^* + \mathbf{E}^*.\boldsymbol{\sigma}.\mathbf{E}\right] \tag{5.265}$$

But for arbitrary matrices and vectors:

$$\mathbf{A}.\mathbf{M}.\mathbf{B} = \mathbf{B}.\mathbf{M}^T.\mathbf{A}; \tag{5.266}$$

(in our dyadic notation we don't explicitly indicate transposes of vectors). So

$$\mathbf{E}.\boldsymbol{\sigma}^*.\mathbf{E}^* = \mathbf{E}^*.\boldsymbol{\sigma}^{*T}.\mathbf{E} \tag{5.267}$$

hence

$$\langle P \rangle = \frac{1}{4}\mathbf{E}^*.\left[\boldsymbol{\sigma}^{*T} + \boldsymbol{\sigma}\right].\mathbf{E} \tag{5.268}$$

If $\boldsymbol{\epsilon} = \mathbf{1} + \frac{1}{-i\omega\epsilon_0}\boldsymbol{\sigma}$ is hermitian $\boldsymbol{\epsilon}^{*T} = \boldsymbol{\epsilon}$, then the conductivity tensor is antihermitian $\boldsymbol{\sigma}^{*T} = -\boldsymbol{\sigma}$ (if ω is real). In that case, equation 5.268 shows that $< P >= 0$. No dissipation. Any dissipation of wave energy is associated with an antihermitian part of $\boldsymbol{\sigma}$ and hence $\boldsymbol{\epsilon}$. Cold Plasma has none.

Collisions introduce damping. Can be included in equation of motion

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right) - m\mathbf{v}\ \nu \tag{5.269}$$

where ν is the collision frequency.

Whole calculation can be followed through replacing $m(-i\omega)$ with $m(\nu - i\omega)$ everywhere. This introduces complex quantity in S, D, P.

We shall not bother with this because in fusion plasmas collisional damping is usually *negligible*. See this physically by saying that transit time of a wave is

$$\frac{\text{Size}}{\text{Speed}} \sim \frac{1\text{ meter}}{3 \times 10^{+8} m/s} \simeq 3 \times 10^{-9}\text{ seconds.} \tag{5.270}$$

(Collision frequency)$^{-1} \sim 10\mu s \to 1ms$, depending on T_e, n_e.

When is the conductivity tensor Antihermitian?

Cold Plasma:

$$\boldsymbol{\epsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \quad \text{where} \quad \begin{array}{c} S = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \Omega_j^2} \\ D = \sum_j \frac{\Omega_j}{\omega} \frac{\omega_{pj}}{\omega^2 - \Omega_j^2} \\ P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \end{array} \tag{5.271}$$

This is manifestly Hermitian *if ω is real*, and then σ is anti-Hermitian.

This observation is sufficient to show that if the plasma is driven with a steady wave, there is no damping, and k does not acquire a complex part.

Two stream Instability

$$\epsilon_{zz} = 1 - \sum_j \frac{\omega_{pj}^2}{(\omega - kv_j)^2} \tag{5.272}$$

In this case, the relevant component is Hermitian (i.e. real) if *both* ω and k are real.

But that just begs the question: If ω and k are real, then there's no damping by definition.

So we can't necessarily detect damping or growth just by inspecting the dieletric tensor form when it depends on *both* ω and k.

Electrostatic Waves in general have $\epsilon = 0$ which is Hermitian. So really it is not enough to deal with ϵ or χ. We need to deal with $\sigma = -i\omega\epsilon_o\chi$, which indeed has a Hermitian component for the two-stream instability (even though χ is Hermitian) because ω is complex.

5.9.8 Ion Acoustic Waves and Landau Damping

We previously derived ion acoustic waves based on fluid treatment giving

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \frac{k^2 p_e \gamma_e}{m_e n_e}} - \frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i n_i}} \tag{5.273}$$

Leading to $\omega^2 \simeq k^2 \left[\frac{\gamma_i T_i + \gamma_e T_e}{m_i}\right]$.

Kinetic treatment adds the extra ingredient of Landau Damping. Vlasov plasma, unmagnetized:

$$\epsilon_{zz} = 1 - \frac{\omega_{pe}^2}{k^2}\int_C \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} - \frac{\omega_{pi}^2}{k^2}\int_C \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oi}}{\partial v}\frac{dv}{n} \tag{5.274}$$

Both electron and ion damping need to be considered as possibly important.

Based on our fluid treatment we know these waves will have *small* phase velocity relative to *electron* thermal speed. Also c_s is somewhat larger than the *ion* thermal speed.

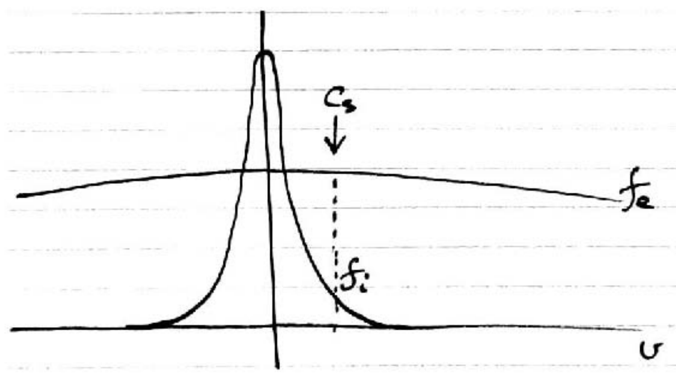

Figure 5.24: Distribution functions of ions and electrons near the sound wave speed.

So we adopt approximations

$$v_{te} \gg \frac{\omega}{k}, \quad v_{ti} < (<)\frac{\omega}{k} \tag{5.275}$$

and expand in opposite ways.

Ions are in the standard limit, so

$$\chi_i \simeq -\frac{\omega_{pi}^2}{\omega^2}\left[1 + \frac{3T_i}{m}\frac{k^2}{\omega^2} + \pi i \frac{\omega^2}{k^2}\frac{1}{n_i}\frac{\partial f_{oi}}{\partial_v}|_{w/k}\right] \tag{5.276}$$

Electrons: we regard $\frac{\omega}{k}$ as small and write

$$\begin{aligned}\wp\int \frac{1}{v-\frac{\omega}{k}}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} &\simeq \wp\int \frac{1}{v}\frac{\partial f_{oe}}{\partial v}\frac{dv}{n} \\ &= \frac{2}{n}\int \frac{\partial f_{oe}}{\partial v^2}dv \\ &= \frac{2}{n}\int -\frac{m_e}{2T_e}f_{oe}dv \qquad \text{for Maxwellian.} \\ &= -\frac{m_e}{T_e}\end{aligned} \tag{5.277}$$

Write $F_0 = f_o/n$.

Contribution from the pole is as usual so

$$\chi_e = -\frac{\omega_{pe}^2}{k^2}\left[-\frac{m_e}{T_e} + \pi i \left.\frac{\partial F_{oe}}{\partial v}\right|_{\omega/k}\right] \tag{5.278}$$

Collecting real and imaginary parts (at real ω)

$$\varepsilon_r(\omega_r) = 1 + \frac{\omega_{pe}^2}{k^2}\frac{m_e}{T_e} - \frac{\omega_{pi}^2}{\omega_r^2}\left[1 + \frac{3T_i}{m}\frac{k^2}{\omega_r^2}\right] \tag{5.279}$$

$$\varepsilon_i(\omega_r) = -\pi\frac{1}{k^2}\left[\omega_{pe}^2\frac{\partial F_{oe}}{\partial v}|_{\omega/k} + \omega_{pi}^2\frac{\partial F_{oi}}{\partial v}|_{\omega/k}\right] \tag{5.280}$$

The real part is essentially the same as before. The extra Bohm Gross term in ions appeared previously in the denominator as

$$\frac{\omega_{pi}^2}{\omega^2 - \frac{k^2 p_i \gamma_i}{m_i}} \leftrightarrow \frac{\omega_{pi}^2}{\omega^2}\left[1 + \frac{3T_i}{m_i}\frac{k^2}{\omega^2}\right] \tag{5.281}$$

Since our kinetic form is based on a rather inaccurate Taylor expansion, it is not clear that it is a better approx. We are probably better off using

$$\frac{\omega_{pi}^2}{\omega^2}\frac{1}{1 - \frac{3T_i k^2}{m_i \omega^2}}. \tag{5.282}$$

Then the solution of $\varepsilon_r(\omega_r) = 0$ is

$$\frac{\omega_r^2}{k^2} = \left[\frac{T_e + 3T_i}{m_i}\right]\frac{1}{1 + k^2\lambda_{De}^2} \tag{5.283}$$

as before, but we've proved that $\gamma_e = 1$ is the correct choice, and kept the $k^2\lambda_{De}^2$ term (1st term of ε_r).

The imaginary part of ε gives damping.

General way to solve for damping when small

We want to solve $\varepsilon(\mathbf{k}, \omega) = 0$ with $\omega = \omega_r + i\omega_i$, ω_i small.

Taylor expand ε about real ω_r:

$$\varepsilon(\omega) \simeq \varepsilon(\omega_r) + i\omega_i \frac{d\varepsilon}{d\omega}|_{\omega_r} \tag{5.284}$$

$$= \varepsilon(\omega_r) + i\omega_i \frac{\partial}{\partial \omega_r}\varepsilon(\omega_r) \tag{5.285}$$

Let ω_r be the solution of $\varepsilon_r(\omega_r) = 0$; then

$$\varepsilon(\omega) = i\varepsilon_i(\omega_r) + i\omega_i \frac{\partial}{\partial \omega_r}\varepsilon(\omega_r). \tag{5.286}$$

This is equal to zero when

$$\omega_i = -\frac{\varepsilon_i(\omega_r)}{\frac{\partial \varepsilon(\omega_r)}{\partial \omega_r}} . \tag{5.287}$$

If, by presumption, $\varepsilon_i \ll \varepsilon_r$, or more precisely (in the vicinity of $\varepsilon = 0$), $\partial\varepsilon_i/\partial\omega_r \ll \partial\varepsilon_r/\partial\omega_r$ then this can be written to lowest order:

$$\omega_i = -\frac{\varepsilon_i\left(\omega_r\right)}{\frac{\partial\varepsilon_r(\omega_r)}{\partial\omega_r}} \tag{5.288}$$

Apply to ion acoustic waves:

$$\frac{\partial\varepsilon_r\left(\omega_r\right)}{\partial\omega_r} = \frac{\omega_{pi}^2}{\omega_r^3}\left[2 + 4\frac{4T_i}{m_i}\frac{k^2}{\omega_r^2}\right] \tag{5.289}$$

so

$$\omega_i = \frac{\pi}{k^2}\frac{\omega_r^3}{\omega_{pi}^2}\left[\frac{1}{2+4\frac{4T_i}{m_i}\frac{k^2}{\omega_r^2}}\right]\left[\omega_{pe}^2\frac{\partial F_{oe}}{\partial v}|_{\omega/k} + \omega_{pi}^2\frac{\partial F_{oi}}{\partial v}|_{\omega/k}\right] \tag{5.290}$$

For Maxwellian distributions, using our previous value for ω_r,

$$\begin{aligned}
\frac{\partial F_{oe}}{\partial v}|_{\frac{\omega_r}{k}} &= \left[-\left(\frac{m_e}{2\pi T_e}\right)^{\frac{1}{2}}\frac{m_e v}{T_e}e^{-\frac{m_e v^2}{2T_e}}\right]_{v=\frac{\omega_r}{k}} \\
&= -\frac{1}{\sqrt{2\pi}}\left(\frac{m_e}{T_e}\right)^{\frac{3}{2}}\left[\frac{T_e+3T_i}{m_i}\right]^{\frac{1}{2}}\frac{1}{\sqrt{1+k^2\lambda_D^2}}\exp\left(-\frac{m_e}{2m_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right) \\
&= -\frac{1}{\sqrt{2\pi}}\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{m_e}{T_e}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_{De}^2}}\,,
\end{aligned} \tag{5.291}$$

where the exponent is of order m_e/m_i here, and so the exponential is 1. And

$$\frac{\partial F_{oi}}{\partial v}|_{\frac{\omega_r}{k}} = -\frac{1}{\sqrt{2\pi}}\frac{m_i}{T_i}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_D^2}}\left(\frac{T_e}{T_i}\right)^{\frac{1}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right) \tag{5.292}$$

Hence

$$\begin{aligned}
\frac{\omega_i}{\omega_r} = &- \frac{\pi}{\sqrt{2\pi}}\frac{\omega_r^2}{k^2}\left[\frac{1}{2+4\frac{3T_i}{m_i}\frac{k^2}{\omega_r^2}}\right]\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{1}{2}}}{\sqrt{1+k^2\lambda_D^2}}\times \\
&\left[\frac{m_i}{m_e}\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}\frac{m_e}{T_e} + \frac{m_i}{T_i}\left(\frac{T_e}{T_i}\right)^{\frac{1}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_D^2}\right)\right]
\end{aligned} \tag{5.293}$$

$$\begin{aligned}
\frac{\omega_i}{\omega_r} = &- \sqrt{\frac{\pi}{2}}\frac{1}{\left[1+k^2\lambda_{De}^2\right]^{\frac{3}{2}}}\frac{\left[1+\frac{3T_i}{T_e}\right]^{\frac{3}{2}}}{2+4\frac{3T_i}{T_e+3T_i}}\times \\
&\left[\underbrace{\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}}_{electron} + \underbrace{\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\exp\left(-\frac{T_e}{2T_i}\frac{1+\frac{3T_i}{T_e}}{1+k^2\lambda_{De}^2}\right)}_{ion\ damping}\right].
\end{aligned} \tag{5.294}$$

[Note: the coefficient on the first line of equation 5.294 for ω_i/ω_r reduces to $\simeq -\sqrt{\pi/8}$ for $T_i/T_e \ll 1$ and $k\lambda_D e \ll 1$.]

Electron Landau damping of ion acoustic waves is rather small: $\frac{\omega_i}{\omega_r} \sim \sqrt{\frac{m_e}{m_i}} \sim \frac{1}{70}$.

Ion Landau damping is *large*, ~ 1 unless the term in the exponent is large. That is

$$\text{unless} \quad \frac{T_e}{T_i} \gg 1 \quad . \tag{5.295}$$

Physics is that large $\frac{T_e}{T_i}$ pulls the phase velocity of the wave: $\sqrt{\frac{T_e+3T_i}{m_i}} = c_s$ above the ion thermal velocity $v_{ti} = \sqrt{\frac{T_i}{m_i}}$. If $c_s \gg v_{ti}$ there are few resonant ions to damp the wave.

[Note. Many texts drop terms of order $\frac{T_i}{T_e}$ early in the treatment, but that is not really accurate. We have kept the first order, giving extra coefficient

$$\left[1+\frac{3T_i}{T_e}\right]^{\frac{3}{2}}\left[\frac{T_e+3T_i}{T_e+6T_i}\right] \simeq 1+\frac{3}{2}\frac{T_i}{T_e} \tag{5.296}$$

and an extra factor $1+\frac{3T_i}{T_e}$ in the exponent. When $T_i \sim T_e$ we ought really to use full solutions based on the Plasma Dispersion Function.]

5.9.9 Alternative expressions of Dielectric Tensor Elements

This subsection gives some useful algebraic relationships that enable one to transform to different expressions sometimes encountered.

$$\begin{aligned}
\chi_{zz} &= \frac{q^2}{\omega m\epsilon_o}\int_C \frac{v\frac{\partial f_o}{\partial v}}{\omega - kv}dv = \frac{q^2}{\omega^2 m\epsilon_o}\frac{\omega}{k}\int_C\left(\frac{\omega}{\omega-kv}-1\right)\frac{\partial f_o}{\partial v}dv && (5.297)\\
&= \frac{q^2}{m\epsilon_o}\frac{1}{k^2}\int_C \frac{1}{\frac{\omega}{k}-v}\frac{\partial f_o}{\partial v}dv && (5.298)\\
&= \frac{\omega_p^2}{k^2}\int_C \frac{1}{\frac{\omega}{k}-v}\frac{1}{n}\frac{\partial f_o}{\partial v}dv && (5.299)\\
&= \frac{\omega_p^2}{k^2}\left[\wp\int \frac{1}{\frac{\omega}{k}-v}\frac{\partial F_o}{2v}dv - \pi i \frac{\partial F_o}{\partial v}|_{\frac{\omega}{k}}\right] && (5.300)
\end{aligned}$$

where $F_o = \frac{f_o}{n}$ is the *normalized* distribution function. Other elements of χ involve integrals of the form

$$\chi_{jl}\frac{\omega m\epsilon_o}{q^2} = \int \frac{v_j\frac{\partial f_o}{\partial v_l}}{\omega - \mathbf{k}.\mathbf{v}}d^3v \ . \tag{5.301}$$

When $\mathbf{k}$ is in z-direction, $\mathbf{k}.\mathbf{v} = k_z v_z$. (Multi dimensional distribution f_0).

If (e.g., χ_{xy}) $l \neq z$ and $j \neq l$ then the integral over v_l yields $\int \frac{\partial f_o}{\partial v_l}dv_l = 0$. If $j = l \neq z$ then

$$\int v_j \frac{\partial f_o}{\partial v_j}dv_j = -\int f_o dv_j \ , \tag{5.302}$$

by parts. So, recalling the definition $f_z \equiv \int f dv_x dv_y$,

$$\begin{aligned} \chi_{xx} = \chi_{yy} &= -\frac{q^2}{\omega m \epsilon_o} \int \frac{f_{oz}}{\omega - \mathbf{k}.\mathbf{v}} dv_z \\ &= -\frac{\omega_p^2}{\omega} \int \frac{F_{oz}}{\omega - \mathbf{k}.\mathbf{v}} dv_z. \end{aligned} \tag{5.303}$$

The fourth type of element is

$$\chi_{xz} = \frac{q^2}{\omega m \epsilon_o} \int \frac{v_x \frac{\partial f_o}{\partial v_z}}{\omega - k_z v_z} d^3 v \ . \tag{5.304}$$

This is not zero unless f_o is isotropic ($= f_o(v)$).
If f is isotropic

$$\frac{\partial f_o}{\partial v_z} = \frac{df_o}{dv} \frac{\partial v}{\partial v_z} = \frac{v_z}{v} \frac{df_o}{dv} \tag{5.305}$$

Then

$$\begin{aligned} \int \frac{v_x \frac{\partial f_o}{\partial v_z}}{\omega - k_z v_z} d^3 v &= \int \frac{v_x v_z}{\omega - k_z v_z} \frac{1}{v} \frac{df_o}{dv} d^3 v \\ &= \int \frac{v_z}{\omega - k_z v_z} \frac{\partial f_o}{\partial v_x} d^3 v = 0 \end{aligned} \tag{5.306}$$

(since the v_x-integral of $\partial f_o / \partial v_x$ is zero). Hence for isotropic $F_o = f_0/n$, with $\mathbf{k}$ in the z-direction,

$$\chi = \begin{bmatrix} -\frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z & 0 & 0^+ \\ 0 & -\frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z & 0^+ \\ 0 & 0 & \frac{\omega_p^2}{k} \int_C \frac{1}{\omega - k v_z} \frac{\partial F_{oz}}{\partial v_z} dv_z \end{bmatrix} \tag{5.307}$$

(and the terms 0^+ are the ones that need isotropy to make them zero).

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_l \end{bmatrix} \tag{5.308}$$

where

$$\epsilon_t = 1 - \frac{\omega_p^2}{\omega} \int_C \frac{F_{oz}}{\omega - k v_z} dv_z \tag{5.309}$$

$$\epsilon_l = 1 - \frac{\omega_p^2}{k^2} \int_C \frac{1}{v - \frac{\omega}{k}} \frac{\partial F_{oz}}{\partial v_z} dv_z \tag{5.310}$$

All integrals are along the Landau contour, passing *below* the pole.

5.9.10 Electromagnetic Waves in unmagnetized Vlasov Plasma

For transverse waves the dispersion relation is

$$\frac{k^2c^2}{\omega^2} = N^2 = \epsilon_t = 1 - \frac{\omega_p^2}{\omega}\frac{1}{n}\int_C \frac{f_{oz}dv_z}{(\omega - k_z v_z)} \tag{5.311}$$

This has, in principle, a contribution from the pole at $\omega - kv_z = 0$. However, for a non-relativistic plasma, thermal velocity is $\ll c$ and the EM wave has phase velocity $\sim c$. Consequently, for all velocities v_z for which f_{oz} is non-zero $kv_z \ll \omega$. We have seen with the cold plasma treatment that the wave phase velocity is actually greater than c. Therefore a proper relativistic distribution function will have no particles at all in resonance with the wave.

Therefore:

1. The imaginary part of ϵ_t from the pole is negligible. And relativisitically zero.

2.

$$\begin{aligned}
\epsilon_t &\simeq 1 - \frac{\omega_p^2}{\omega^2}\frac{1}{n}\int_{-\infty}^{\infty} f_{oz}\left(1 + \frac{kv_z}{\omega} + \frac{k^2v_z^2}{\omega^2} + ...\right) dv_z \\
&= 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{k^2}{\omega^2}\frac{T}{m} + ...\right] \\
&\simeq 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{k^2v_t^2}{\omega^2}\right] \\
&\simeq 1 - \frac{\omega_p^2}{\omega^2}
\end{aligned} \tag{5.312}$$

Thermal correction to the refractive index N is small because $\frac{k^2v_t^2}{\omega^2} \ll 1$.

Electromagnetic waves are hardly affected by Kinetic Theory treatment in unmagnetized plasma. Cold Plasma treatment is generally good enough.