Lecture 16: Variational Principle

Variational Principle

$$\omega^2 = \frac{\delta W}{K}$$

$$\delta W = -\frac{1}{2}\int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r}$$

$$K = \frac{1}{2}\int \rho\left|\xi\right|^2 dr$$

Advantages:

1. allows use of trial functions to estimate ω^2
2. can be applied to multidimensional systems efficiently

Disadvantages:

1. still somewhat complicated
2. gives more information than minimum required

Energy Principle

1. Sometimes we only want to know whether the system is stable or not
2. No great need to know eigenvalues ω^2
3. Growth rate are very fast $r^2 \sim v_T^2/a^2 \sim$ (10 μsec)
4. Experimental times $\sim$ 10 msec – sec's.
5. Since MHD instabilities are very strong, it is more important to know whether system is stable or not, rather than know the precise growth rate (which can be easily estimated)
6. In these cases, the variational principle can be simplified further, yielding the Energy Principle. This is a simpler variational procedure which accurately gives <u>stability boundaries</u> but only estimates growth rates.

The Energy Principle

1. Variational Principle $\qquad \omega^2 = \dfrac{\delta W}{K}$

If all $\omega^2 > 0$, the system is stable

2. Energy Principle $\delta W \geq 0$ for all allowable displacements, the system is stable.
   Any displacement which makes $\delta W < 0 \Rightarrow$ instability

Proof: (based on normal modes) more general proof in text book

1. Assume complete set of normal modes, orthonormal

$$-\omega_n^2 \rho \underline{\xi}_n = \underline{F}\left(\underline{\xi}_n\right) \qquad \int \rho \underline{\xi}_H^* \cdot \xi_{\underline{m}} d\underline{r} = \xi_{mn}$$

2. Arbitrary trial function

$$\underline{\xi} = \sum a_n \underline{\xi}_n$$

3. Evaluate δW

$$\delta W = -\frac{1}{2}\int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) = -\frac{1}{2}\sum a_n^* a_m \int \underline{\xi}_n^* \cdot \underline{F}\left(\underline{\xi}_m\right) d\underline{r}$$

$$= -\frac{1}{2}\sum a_n^* a_m \int \underline{\xi}_n^* \cdot \left(-\omega_m^2 \rho \underline{\xi}_m\right)$$

$$= \frac{1}{2}\sum \omega_m^2 \left|a_m\right|^2$$

4. If a trial function can be found which makes $\delta W < 0$, then at least one $\omega_m^2 < 0 \rightarrow$ instability

5. If all trial function make $\delta W > 0$, then all $\omega_m^2 > 0 \rightarrow$ stability

Extended Energy Principle

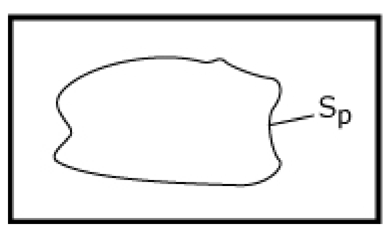
$$\delta W = -\frac{1}{2}\int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r}$$

$$\underline{F}\left(\underline{\xi}\right) = \underline{J}_1 \times \underline{B}_0 + \underline{J}_0 \times \underline{B}_1 - \nabla p_1$$

$$= \left[\nabla \times \nabla \times \left(\underline{\xi} \times \underline{B}\right)\right] \times \underline{B} + \left(\nabla \times \underline{B}\right) \times \left[\nabla \times \left(\underline{\xi} \times \underline{B}\right)\right] \times \nabla\left[\underline{\xi} \cdot \nabla p + rp \nabla \cdot \underline{\xi}\right]$$

1. Valid with wall on plasma

$$\underline{n} \cdot \underline{\xi}\Big|_{S_p} = 0$$

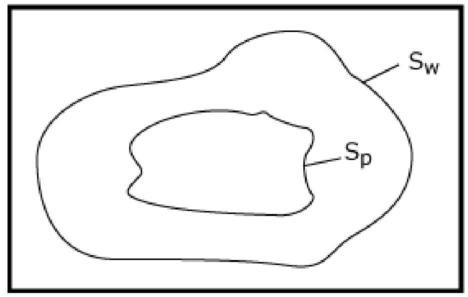


2. Valid with vacuum region

$[\![\underline{n}\cdot\underline{B}]\!]_{S_p} = 0$

$\left[\!\!\left[p+\frac{B^2}{2\mu_0}\right]\!\!\right]_{S_p} = 0$

$\underline{n}\cdot\underline{B}_1\big|_{S_w} = 0$



3. δW above not convenient because of complicated boundary condition with wall, and no explicit appearance of Vacuum energy.

4. These are resolved by Extended Energy Principle

Extended Energy Principle

1. Rewrite δW_1 introduce natural boundary conditions

2. $$\delta W = -\frac{1}{2}\int d\underline{r}\left[\underbrace{\nabla\times\nabla\times\left(\underline{\xi}\times\underline{B}\right)\Big]\times\frac{\underline{B}}{\mu_0}}_{\text{integrate by parts}}+\frac{\nabla\times B}{\mu_0}\times\left[\nabla\times\xi\times B\right]+\nabla\left(\underline{\xi}\cdot\nabla p+\underbrace{rp\nabla\cdot\underline{\xi}}_{\text{integrate by parts}}\right)\right]\cdot\underline{\xi}^*$$

3. Define $\underline{Q}\equiv\underline{B}_1=\nabla\times\left(\underline{\xi}\times\underline{B}\right)$

$$\delta W = +\frac{1}{2}\int d\underline{r}\left\{\frac{\left|\underline{Q}\right|^2}{\mu_0}+rp\left|\nabla\cdot\underline{\xi}\right|^2-\underline{\xi}^*\cdot\left[\underline{J}\times\underline{Q}+\nabla\left(\underline{\xi}\cdot\nabla p\right)\right]\right\}-\frac{1}{2}\int ds\left(\underline{n}\cdot\underline{\xi}^*\right)\left[rp\nabla\cdot\underline{\xi}-\frac{\underline{B}\cdot\underline{B}_1}{\mu_0}\right]$$

4. Separate $\xi_\perp,\xi_\parallel$: $\underline{\xi}=\underline{\xi_\perp}+\xi_\parallel\underline{b}$

5. It is easily shown that $\underline{b}\cdot\left[\underline{J}\times\underline{Q}+\nabla\left(\underline{\xi}\cdot\nabla p\right)\right]=0$ so that last term becomes

$$\underline{\xi_\perp^*}\cdot\left[\underline{J}\times\underline{Q}+\nabla\left(\underline{\xi}\cdot\nabla p\right)\right]$$

integrate by parts

note $\underline{Q}=\nabla\times\left(\underline{\xi}\times\underline{B}\right)=\nabla\times\left(\underline{\xi_\perp}\times\underline{B}\right)$

$\underline{\xi}\cdot\nabla p=\underline{\xi_\perp}\cdot\nabla p$

6. $\delta W = \delta W_F + B.T.$

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left[\frac{|\underline{Q}|^2}{\mu_0} - \underline{\xi}_\perp^* \cdot \underline{J}\times\underline{Q} + rp\left|\nabla\cdot\underline{\xi}\right|^2 + \left|\underline{\xi}_\perp\cdot\nabla p\right|\nabla\cdot\underline{\xi}_\perp^*\right]$$

Standard form of the fluid energy

$$BT = \frac{1}{2}\int dS\ \underline{n}\cdot\underline{\xi}_\perp^*\left[\frac{\underline{B}\cdot\underline{B}_1}{\mu_0} - rp\,\nabla\cdot\underline{\xi} - \underline{\xi}_\perp\cdot\nabla p\right]$$

7. Introduce natural boundary condition

$$\left[\!\left[p + \frac{B^2}{2\mu_0}\right]\!\right] = 0 \qquad \text{linearize}$$

$$\left[\underbrace{p_1}_{-rp\nabla\cdot\underline{\xi} - \underline{\xi}_\perp\cdot\nabla p} + \frac{B\cdot B_1}{\mu_0} + \underline{\xi}\cdot\nabla\left(p + \frac{B^2}{2\mu_0}\right)\right]_{S_p} = \left[\frac{\hat{B}\cdot\hat{B}_1}{\mu_0} + \underline{\xi}\cdot\nabla\frac{\hat{B}^2}{2\mu_0}\right]$$

8. Substitute above

$$BT = \delta W_s + \frac{1}{2}\int\underline{n}\cdot\underline{\xi}^*\,\frac{\left(\hat{\underline{B}}\cdot\hat{\underline{B}}_1\right)}{\mu_0}dS$$

$$\delta W_s = \frac{1}{2}\int dS\left(\underline{n}\cdot\underline{\xi}^*\right)\left[\!\left[\underline{\xi}\cdot\nabla\left(p + \frac{B^2}{2\mu_0}\right)\right]\!\right] \qquad \underline{\xi} = \xi_n\underline{n} + \xi_{||}\underline{b} + \xi_\perp\underline{b}\times\underline{n}$$

only term which contribute

$$= \frac{1}{2}\int dS\left|\underline{n}\cdot\underline{\xi}\right|^2 n\cdot\left[\!\left[\nabla\left(p + \frac{B^2}{2\mu_0}\right)\right]\!\right]$$

Surface term is non-zero only if surface currents flow

9. $\delta W = \delta W_F + \delta W_s + \dfrac{1}{2}\displaystyle\int dS\,\underline{n}\cdot\underline{\xi}^*\,\frac{\hat{\underline{B}}\cdot\hat{\underline{B}}_1}{\mu_0}$

10. Show last term is related to vacuum energy

11. $\delta W_v = \dfrac{1}{2\mu_0}\displaystyle\int_v\left|\hat{\underline{B}}_1^2\right|d\underline{r} = \dfrac{1}{2\mu_0}\int\left|\nabla\times\underline{A}_1\right|^2 d\underline{r}$

$$= \frac{1}{2\mu_0}\int d\underline{r}\left[\nabla\cdot\left(\underline{A}_1^*\times\nabla\times\underline{A}_1\right) - \underline{A}_1^*\cdot\underbrace{\nabla\times\nabla\times\underline{A}_1}_{=0}\right]$$

$$= -\frac{1}{2\mu_0}\int_S dS\,\underline{n}\cdot\left(\hat{\underline{A}}_1^* \times \hat{\underline{B}}_1\right)$$

12. But: since $\underline{n}\cdot\hat{\underline{B}}_1 = \underline{n}\cdot\nabla\times\hat{\underline{A}}_1 = \underline{n}\cdot\nabla\times\underline{\xi}\times\hat{\underline{B}}$

then $\hat{\underline{A}}_1 = \underline{\xi_\perp}\times\underline{B}+\nabla\phi$ Choose $\hat{\underline{B}}_1\cdot\left(\underline{n}\times\nabla\phi\right)=0$ as gauge

and $\delta W_v = -\frac{1}{2\mu_0}\int dS\;\underline{n}\cdot\left(\underline{\xi_\perp^*}\times\underline{B}\right)\times\hat{\underline{B}}_1$

$$= \frac{1}{2\mu_0}\int dS\;\underline{n}\cdot\underline{\xi}^*\,\hat{\underline{B}}\cdot\hat{\underline{B}}_1$$

Extended Energy Principle

$\delta W = \delta W_F + \delta W_S + \delta W_V$

Boundary Conditions on trial functions

$\underline{n}\cdot\hat{\underline{B}}_1\Big|_{S_w} = 0$

$\underline{n}\cdot\hat{\underline{B}}_1\Big|_{S_p} = \underline{n}\cdot\underline{B}_1\Big|_{S_p} = \underline{n}\cdot\nabla\times\left(\underline{\xi}\times\underline{B}\right)\Big|_{S_p}$

$$= -\left(\underline{n}\cdot\underline{\xi}\right)\left[\underline{n}\cdot\left(\underline{n}\cdot\nabla\right)\underline{B}\right]+\underline{B}\cdot\nabla\left(\underline{n}\cdot\underline{\xi}\right)\Big|_{S_p}$$

depends only on $\underline{n}\cdot\underline{\xi}$

pressure balance conditions not required $\rightarrow$ natural boundary conditions

Final Step

Intuitive form of δW_F

1. Standard form OK

2. Intuitive form gives more insight.

3. $\delta W_F = \frac{1}{2}\int d\underline{r}\left\{\frac{\left|\underline{Q}\right|^2}{\mu_0} - \underline{\xi_\perp^*} - \underline{J}\times\underline{Q} + r p\left|\nabla\cdot\underline{\xi}\right|^2 + \left(\underline{\xi_\perp}\cdot\nabla p\right)\nabla\cdot\underline{\xi_\perp^*}\right\}$

$\left|\underline{Q}\right|^2 = \left|\underline{Q_\perp}\right|^2 + \left|Q_{||}\right|^2$

$$\underline{\xi}_\perp^* \cdot \underline{J} \times \underline{Q} = \left(\underline{\xi}_\perp^* \times \underline{b}\right) \cdot \underline{Q}_\perp J_\| + Q_\| \underline{\xi}_\perp^* \cdot \underline{J}_\perp \times \underline{b}$$

now: $$\underline{J}_\perp = \frac{\underline{b} \times \nabla p}{B} \qquad \left(\underline{J} \times \underline{B} = \nabla p\right)$$

$$Q_\| = \underline{b} \cdot \nabla \times \left(\underline{\xi}_\perp \times \underline{B}\right)$$

$$= \underline{b} \cdot \left(\underline{B} \cdot \nabla \underline{\xi}_\perp - \underline{\xi}_\perp \cdot \nabla \underline{B} - \underline{B} \nabla \cdot \underline{\xi}_\perp\right)$$

$$= -B\left(\nabla \cdot \underline{\xi}_\perp + 2\underline{\xi}_\perp \cdot \underline{\kappa}\right) + \frac{\mu_0}{B} \underline{\xi}_\perp \cdot \nabla p$$

Substitute back

$$\delta W_F = \frac{1}{2}\int d\underline{r}\left[\overbrace{\frac{\left|\underline{Q}_\perp\right|^2}{\mu_0}}^{1} + \overbrace{\frac{B^2}{\mu_0}\left|\nabla \cdot \underline{\xi}_\perp + 2\underline{\xi}_\perp \cdot \underline{\kappa}\right|^2}^{2} + \overbrace{\gamma p\left|\nabla \cdot \underline{\xi}\right|^2}^{3} - \overbrace{2\left(\underline{\xi}_\perp \cdot \nabla p\right)\left(\underline{\kappa} \cdot \underline{\xi}_\perp^*\right)}^{4} - \overbrace{J_\| \left(\underline{\xi}_\perp^* \times \underline{b}\right) \cdot \underline{Q}_\perp}^{5}\right]$$

1. line bending > 0 — shear alform wave
2. magnetic compression > 0 — compressional alform wave
3. plasma compression > 0 — sound wave
4. pressure driven modes + or –
5. current driven modes (kinks) + or –

Summary

Energy Principle: $\delta W = \delta W_F + \delta W_S + \delta W_V$

$\delta W \geq 0$ for all allowable displacements $\rightarrow$ stability

$\delta W < 0$ for any allowable displacement $\rightarrow$ instability

Minimize δW with respect to three components of $\underline{\xi}$.

Incompressibility

1. Because of the simple way in which $\xi_\|$ appears in δW, it is possible to minimize once for all with respect to $\xi_\|$ and eliminate it from the calculation.

2. Only appearance of $\xi_{\parallel}$

$$\delta W_{\parallel} = \int d\underline{r} \;\; rp\left|\nabla \cdot \underline{\xi}\right|^2$$

3. Let $\xi_{\parallel} \to \xi_{\parallel} + \delta\xi_{\parallel}$

4. Vary $\delta W_{\parallel}$ $\quad \underline{\xi} = \underline{\xi_{\perp}} + \xi_{\parallel}\frac{\underline{B}}{B}$

$$\delta\left(\delta W_{\parallel}\right) = \int d\underline{r} \;\; rp\left(\nabla \cdot \underline{\xi}\right)\nabla \cdot \left(\delta\xi_{\parallel}\frac{\underline{B}}{B}\right)$$

integrate by parts

$$= -\int d\underline{r}\,\frac{\delta\xi_{\parallel}}{B}\,\underline{B} \cdot \nabla\left(rp\,\nabla \cdot \underline{\xi}\right)$$

$$= -\int d\underline{r}\,\frac{\delta\xi_{\parallel}}{B}\,rp\underline{B} \cdot \nabla\left(\nabla \cdot \underline{\xi}\right)$$

5. Several minimizing condition

$$\boxed{\underline{B} \cdot \nabla\left(\nabla \cdot \underline{\xi}\right) = 0}$$

6. If $\underline{B} \cdot \nabla$ is non-singular then

$\nabla \cdot \underline{\xi} = 0$ (obvious)

$$\delta W_{\parallel} = \frac{1}{2}\int rp\left|\nabla \cdot \underline{\xi}\right|^2 \to 0$$

7. Two cases where $\nabla \cdot \underline{\xi}$ cannot be set to zero

8. Special symmetry

Example: Z pinch $\quad \underline{B} = B_{\theta}\left(r\right)\underline{e}_{\theta} \quad \underline{\xi} \sim e^{im\theta + ikz}\underline{\xi}\left(r\right)$

$$\underline{B} \cdot \nabla\frac{\xi_{\parallel}}{B} = \frac{B_{\theta}}{r}\frac{\partial}{\partial\theta}\frac{\xi_{\parallel}}{B_{\theta}} = \frac{im\xi_{\parallel}}{r}$$

For $m = 0 \quad \underline{B} \cdot \nabla\frac{\xi_{\parallel}}{B} = 0$

Note: $\nabla \cdot \underline{\xi} = \nabla \cdot \underline{\xi_{\perp}} + \nabla \cdot \frac{\xi_{\parallel}}{B}\underline{B} = \nabla \cdot \underline{\xi_{\perp}} + \underline{B} \cdot \nabla\frac{\xi_{\parallel}}{B}$

In special symmetry $\nabla \cdot \underline{\xi} = \nabla \cdot \underline{\xi_\perp}$ and ξ_n does not appear. The term $rp\left|\nabla \cdot \underline{\xi_\perp}\right|^2$ must be maintained for the rest of the minimization.

9. Closed line (periodicity constraints). Choose $\xi_{||}$ so $\nabla \cdot \xi = 0$

$$\underline{B} \cdot \nabla \frac{\xi_{||}}{B} = -\nabla \cdot \underline{\xi_\perp} = B\frac{\partial \xi_{||}/B}{\partial l}$$

$$\frac{\xi_{||}}{B} = -\int \frac{\nabla \cdot \xi_\perp}{B} dl$$

In general $\frac{\xi_{||}}{B}(l + L) \neq \frac{\xi_{||}}{B}(l) \rightarrow$ no periodicity

Solution

$\underline{B} \cdot \nabla \quad \nabla \cdot \underline{\xi} = 0$

$$\therefore \nabla \cdot \underline{\xi} = F(p)$$

homogenous solution

$$\underline{B} \cdot \nabla \frac{\xi_{||}}{B} = -\nabla \cdot \underline{\xi_\perp} + F(p)$$

$$\frac{\xi_{||}}{B} = -\int \frac{\nabla \cdot \xi_\perp}{B} dl + \int \frac{F(p)\, dl}{B} = -\int_0^l \frac{\nabla \cdot \xi_\perp}{B} dl + F(p)\int_0^l \frac{dl}{B}$$

In periodicity choose

$$F(p) = < \nabla \cdot \underline{\xi_\perp} > = \frac{\oint \frac{dl}{B} \nabla \cdot \underline{\xi_\perp}}{\oint \frac{dl}{B}}$$

Then $\delta W_{||} = \frac{1}{2}\int rp\left|\nabla \cdot \underline{\xi}\right|^2 d\underline{r} = \frac{1}{2}\int rp F^2\, d\underline{r}$

$$\boxed{\delta W_{||} = \frac{1}{2}\int d\underline{r}\, rp\left|< \nabla \cdot \underline{\xi_\perp} >\right|^2}$$

Only a function of $\xi_\perp$

Summary of internal modes in a straight tokamak

1. $m \geq 2$ stable
2. $m = 1,\ n = 1$ must use for $n = 1$, requires $q(0) > 1$ for stability
3. internal modes do not limit β, or I $\left(q(a)\right)$, but clamp $q(0) \approx 1$, by sawtooth oscillations
4. To show instability we needed to calculate $\delta W = e^2 \underset{\substack{\parallel \\ 0}}{\delta W_2} + \in^4 \underset{\text{must}}{\delta W_4}$

Consider now external modes

1. Vacuum no force free fields
2. $m = 1$ Kruskal Shafranov limit
3. High m external kinks