22.616 Plasma Transport Theory

Problem # 2 Solutions

1. Equilibrium: Find in text book:

$$\nu_S^{ab}(v) = \hat{\nu}_{ab} \frac{2T_a}{T_b}\left(1+\frac{m_b}{m_a}\right)\frac{G(x_b)}{x_a}$$

$$\nu_\parallel^{ab}(v) = 2\hat{\nu}_{ab}\frac{G(x_b)}{x_a^3}$$

where $\quad x_a \equiv \frac{v}{v_{Ta}}, \quad x_b \equiv \frac{v}{v_{Tb}}, \quad v_{Ta} \equiv \sqrt{\frac{2T_a}{m_a}}$

For like particles, $a=b$, let $x = \frac{v}{v_T}$. $v_T \equiv \cancel{v_a} v_{Ta} = v_{Tb}$

Then $T_a = T_{\cancel{a}b} \equiv T$, we can easily find

$$\frac{\nu_S^{ab}(v)}{\nu_\parallel^{ab}(v)} = \left(1+\frac{m_{\cancel{a}b}}{m_{\cancel{b}a}}\right)x_a^2 = 2x^2 = 2\frac{v^2}{v_T^2}$$

i.e., $$\frac{\nu_S^{ab}(v)}{v\,\nu_\parallel^{ab}(v)} = 2\frac{v}{v_T^2}$$

From Eq. (3.40), we find the velocity magnitude part of the collisional operator:

$$C_v = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^3\frac{m_{ab}}{m_a+m_b}\nu_S^{ab} f_a + \frac{1}{2}\nu_\parallel^{ab} v \frac{\partial f_a}{\partial v}\right)$$

$$= \frac{1}{2}\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^4\nu_\parallel^{ab}\left(\frac{\nu_S^{ab}}{v\,\nu_\parallel^{ab}} f_a + \frac{\partial f_a}{\partial v}\right)\right)$$

$$= \frac{1}{2v^2}\frac{\partial}{\partial v}\left[v^4\nu_\parallel^{ab}\left(2\frac{v}{v_T^2}f + \frac{\partial f}{\partial v}\right)\right] \quad (f \equiv f_a)$$

Therefore $\nu_{\parallel}(v) = \nu_{\parallel}^{ab}(v)$

$$= 2\hat{\nu}_{aa}\frac{G(x_a)}{x_a^3} = 2\hat{\nu}_{aa} v_T^2 \frac{G(x)}{v^3}$$

when $G_v \to 0$

$$v^4 \nu_{\parallel}(v)\left(2\frac{v}{v_T^2}f + \frac{\partial f}{\partial v}\right) = const$$

Notice $v^4\nu_{\parallel}(v) \propto G(x)v$ & $G(0) = 0$

So when $v \to 0$, $v^4\nu_{\parallel}(v) = 0$, then $const = 0$, i.e

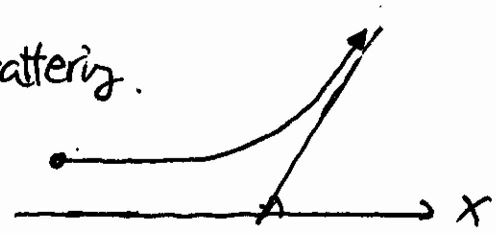
$$2\frac{v}{v_T}f + \frac{\partial f}{\partial v} = 0$$

Solve it we get: $f = \frac{1}{\sqrt{\pi v_T^2}} e^{-\frac{v^2}{v_T^2}} = f_M$

2. Fokker-Planck equation accuracy

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \underline{v}}\cdot \underline{A} f + \frac{\partial^2}{\partial \underline{v}\partial \underline{v}} : \underline{\underline{D}} f + \frac{\partial^3}{\partial \underline{v}\partial \underline{v}\partial \underline{v}} \vdots \underline{\underline{I}} f$$

Use the notation in the textbook; In an orthogonal coordinate system (x, y, z), with x in the direct of the incident velocity of test particle. For small angle scattering.



$$\Delta V_x = -\left(1+\frac{m_a}{m_b}\right)\left(\frac{e_a e_b}{2\pi\epsilon_0 m_a}\right)^2 \frac{1}{2r^2u^3}$$

$$\Delta V_y \simeq \frac{e_a e_b}{2\pi\epsilon_0 m_a}\frac{\cos\phi}{ur}$$

$$\Delta V_z \simeq \frac{e_a e_b}{2\pi\epsilon_0 m_a}\frac{\sin\phi}{ur}$$

with $\quad u = |\underline{v} - \underline{v}'|$

First consider tensor $\underline{\underline{T}}$.

$$\text{(I)}\quad T_{ijk} = \left\langle \frac{\Delta V_i \Delta V_j \Delta V_k}{\Delta t}\right\rangle \frac{1}{6}\quad , \; i,j,k = x, y, z$$

$$= \frac{1}{6}\int d^3v' dr\, r d\phi\, f_b(v')\, u \Delta V_i \Delta V_j \Delta V_k$$

Since $\Delta V_x \propto \frac{1}{r^2}$, $\Delta V_y, \Delta V_z \propto \frac{1}{r}$. So $\Delta V_i \Delta V_j \Delta V_k$ at ~~most~~ least gives $\frac{1}{r^3}$, then the integral $\int_{r_{min}}^{\lambda_D} \frac{1}{r^3} r dr \simeq \frac{1}{r_{min}}$

Also. we notice $\int_0^{2\pi} d\phi \sin\phi = \int_0^{2\pi} d\phi \cos\phi = \int_0^{2\pi} d\phi \sin^3\phi = \int_0^{2\pi} d\phi \cos^3\phi$

$= \int_0^{2\pi} d\phi \sin^2\phi \cos\phi = \int_0^{2\pi} d\phi \cos^2\phi \sin\phi = 0$. Then only $\Delta V_x \Delta V_y^2$ $\Delta V_x \Delta V_z^2$, ΔV_x^3 give ~~nonvanishing~~ non-vanishing terms. These terms at least give $\frac{1}{r^4}$. So there is no divergent term $\ln\Lambda$ in the tensor of T_{ijk}. For the same reasoning. this is true for even higher order terms than $\underline{\underline{T}}$.

From this point of view, the Fokker-Planck equation has an inherent error of $\frac{1}{\ln\Lambda} \sim 3\%$ for Fusion Plasma.

3. Collision Operator Properties:

$$C_{ab}(f_a, f_b) = \frac{1}{2}\Gamma^{ab}\frac{\partial}{\partial \underline{v}}\cdot\int d^3v' \,\underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(\frac{\partial}{\partial \underline{v}} - \frac{m_a}{m_b}\frac{\partial}{\partial \underline{v}'}\right)f_a(\underline{v})f_b(\underline{v}')$$

with $\Gamma^{ab} \equiv \dfrac{4\pi z_a^2 z_b^2 e^4 \ln\Lambda}{m_a^2}$

Define $\Gamma \equiv \dfrac{4\pi z_a^2 z_b^2 e^4 \ln\Lambda}{m_a m_b}$, we first prove the two-species case, then use the two-species result, write down the single-species result directly.

Two species-case

1°. particle conservation

$$\int C_{ei}(f_e, f_i)\, d^3v$$

$$= \frac{1}{2}\Gamma^{ei} \int d^3v \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Gauss' law

$$= \frac{1}{2}\Gamma^{ei} \oint_{\infty} d\vec{S} \cdot \int d^3v' \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Because $f_e(\underline{v})$, $f_i(\underline{v}')$, $\frac{\partial}{\partial \underline{v}} f_e$, $\frac{\partial}{\partial \underline{v}'} f_i$ vanishes at infinity

$$\underline{\int C_{ei}(f_e, f_i)\, d^3v = 0}$$

For the same reason $\int C_{ie}(f_i, f_e)\, d^3v = 0$

$\int C_{ee}(f_e, f_e)\, d^3v = 0$, $\int C_{ii}(f_i, f_i)\, d^3v = 0$

2° Momentum Conservation.

$\int C_{ei}$: The momentum increase of electron due to colliding with ions

$$\int C_{ei}(f_e, f_i)\, m_e \underline{v}\, d^3v$$

$$= \frac{1}{2}\Gamma^{ei} \int d^3v\, m_e \underline{v} \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Integrate by parts

$$= -\frac{m_e}{2}\Gamma^{ei} \int d^3v\, d^3v' \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

$$= \frac{\Gamma}{2}\int d^3v\, d^3v'\, \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_e \frac{\partial}{\partial \underline{v}'} - m_i \frac{\partial}{\partial \underline{v}}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Similarly

$$\int C_{ie}(f_i, f_e)\, m_i \underline{v}\, d^3v$$

$$= \frac{\Gamma}{2}\int d^3v\, d^3v'\, \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_i \frac{\partial}{\partial \underline{v}'} - m_e \frac{\partial}{\partial \underline{v}}\right) f_i(\underline{v}) f_e(\underline{v}')$$

Change of variables, $\underline{v} \rightleftarrows v'$, Note $\underline{\underline{U}}(\underline{v}-\underline{v}') = \underline{\underline{U}}(\underline{v}'-\underline{v})$

$$\int C_{ie}(f_i, f_e)\, m_i \underline{v}\, d^3v$$

$$= -\frac{\Gamma}{2}\int d^3v\, d^3v'\, \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_e \frac{\partial}{\partial \underline{v}'} - m_i \frac{\partial}{\partial \underline{v}}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Therefore

$$\int d^3v \left(C_{ei}(f_e, f_i)\, m_e \underline{v} + C_{ie}(f_i, f_e)\, m_i \underline{v} \right) = 0$$

~~i.e. the net transfer of momentum between electrons & ions are~~ is

~~Zero, momentum~~ is conserved.

i.e. All the momentum electron lost (gain) is equal to

the momentum ions gain (lost), Momentum is conserved.

3°. Energy Conservation

$$\int C_{ei}(f_e, f_i)\frac{m_e v^2}{2}d^3v$$

$$= \frac{1}{4}\Gamma^{ei}\int d^3v\, m_e v^2 \frac{\partial}{\partial \underline{v}}\cdot \int d^3v'\, \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(\frac{\partial}{\partial \underline{v}} - \frac{m_e}{m_i}\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

$$= \frac{1}{4}\Gamma \int d^3v\, v^2 \frac{\partial}{\partial \underline{v}}\cdot \int d^3v'\, \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_i\frac{\partial}{\partial \underline{v}} - m_e\frac{\partial}{\partial \underline{v}'}\right) f_e(\underline{v}) f_i(\underline{v}')$$

integrate by parts

$$= \frac{1}{2}\Gamma \int d^3v\, d^3v'\, \underline{v}\cdot \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_e\frac{\partial}{\partial \underline{v}'} - m_i\frac{\partial}{\partial \underline{v}}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Similarly

$$\int C_{ie}(f_i, f_e)\frac{m_i v^2}{2}d^3v$$

$$= \frac{1}{2}\Gamma \int d^3v\, d^3v'\, \underline{v}\cdot \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_i\frac{\partial}{\partial \underline{v}'} - m_e\frac{\partial}{\partial \underline{v}}\right) f_i(\underline{v}) f_e(\underline{v}')$$

change of varibles $\underline{v} \Leftrightarrow \underline{v}'$

$$= \frac{\Gamma}{2}\int d^3v\, d^3v'\, \underline{v}'\cdot \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_i\frac{\partial}{\partial \underline{v}} - m_e\frac{\partial}{\partial \underline{v}'}\right)\cdot f_e(\underline{v}) f_i(\underline{v}')$$

Then $\int\left(C_{ie}(f_i, f_e)\frac{m_i v^2}{2} + C_{ei}(f_e, f_i)\frac{m_e v^2}{2}\right)d^3v$

$$= \frac{\Gamma}{2}\int d^3v\, d^3v'\, (\underline{v}-\underline{v}')\cdot \underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(m_e\frac{\partial}{\partial \underline{v}'} - m_i\frac{\partial}{\partial \underline{v}}\right) f_e(\underline{v}) f_i(\underline{v}')$$

Because $(\underline{v}-\underline{v}')\cdot\underline{\underline{U}}(\underline{v}-\underline{v}')=0$, So

$$\int d^3v\left(C_{ei}(f_e,f_i)\frac{m_e v^2}{2}+C_{ie}(f_i,f_e)\frac{m_i v^2}{2}\right)=0$$

This means, ~~the~~ energe is conserved.

single species case:

1° Particle conservation: proved in the two-species case.

2° Momentum Conservation: Use the result from two-species case

$$\int C_{aa}(f_a,f_a)m_a\underline{v}\,d^3v$$

$$=-m\frac{\Gamma}{2}\int\underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(\frac{\partial}{\partial\underline{v}}-\frac{\partial}{\partial\underline{v}'}\right)f_a(\underline{v})f_a(\underline{v}')d^3v'd^3v$$

$\underline{v}\Leftrightarrow\underline{v}'$

$$=-m\frac{\Gamma}{2}\int\underline{\underline{U}}(\underline{v}-\underline{v}')\cdot\left(\frac{\partial}{\partial\underline{v}'}-\frac{\partial}{\partial\underline{v}}\right)f_a(\underline{v})f_a(\underline{v}')d^3v'd^3v$$

$$=-\int C_{aa}(f_a,f_a)m_a\underline{v}\,d^3v$$

So $\int C_{aa}(f_a,f_a)m_a\underline{v}\,d^3v=0$

3°. Energy Conservation: use the intermediate result from two-species case

$$\int C_{aa}(f_a, f_a)\, \tfrac{1}{2} m_a v^2 \, d^3v$$

$$= \frac{m_a}{2}\Gamma \int d^3v\, d^3v'\, \underline{v} \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}}\right) f_a(\underline{v})\, f_a(\underline{v}') \qquad ①$$

$$\overset{\underline{v} \Leftrightarrow \underline{v}'}{=} \frac{m_a}{2}\Gamma \int d^3v\, d^3v'\, \underline{v}' \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'}\right) f_a(\underline{v}) f_a(\underline{v}') \qquad ②$$

$$\overset{\frac{①+②}{2}}{=} \frac{m_a}{4}\Gamma \int d^3v\, d^3v'\, (\underline{v}-\underline{v}') \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}}\right) f_a(\underline{v}) f_a(\underline{v}')$$

$$= 0$$

4. H-Theorem: (single species)

$$S \equiv -f \ln f$$

Then $\dfrac{dS}{dt} = -\dfrac{d}{dt} \int d^3v\, f \ln f$

$$= -\int d^3v \left(\frac{\partial f}{\partial t} \ln f + \frac{\partial f}{\partial t}\right)$$

$$= -\int d^3v\, C(f,f) \ln f - \underbrace{\int d^3v\, C(f,f)}_{=0 \text{ particle conserving}}$$

$$= -\int d^3v \ln f\, C(f,f)$$

Plug into the Landau Collisional Operator

$$\frac{dS}{dt} = -\frac{1}{2}\Gamma \int d^3v \ln f \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'}\right) f f'$$

Integrate by parts

$$\cancel{= \frac{1}{2}\Gamma \int d^3v \left(\frac{\partial}{\partial \underline{v}} \ln f\right) \cdot \int d^3v' \underline{\underline{U}}}$$

$$= \frac{1}{2}\Gamma \int d^3v\, d^3v' \left(\frac{\partial}{\partial \underline{v}} \ln f\right) \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'}\right) f f' \quad ①$$

$\underline{v} \Leftrightarrow \underline{v}'$

$$= \frac{1}{2}\Gamma \int d^3v\, d^3v' \left(\frac{\partial}{\partial \underline{v}'} \ln f'\right) \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}}\right) f f' \quad ②$$

$$\frac{①+②}{2} = \frac{1}{4}\Gamma \int d^3v\, d^3v' \left(\frac{\partial}{\partial \underline{v}} \ln f - \frac{\partial}{\partial \underline{v}} \ln f'\right) \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'}\right) f f'$$

$$= \frac{1}{4}\Gamma \int d^3v\, d^3v' f f' \left(\frac{\partial}{\partial \underline{v}} \ln f - \frac{\partial}{\partial \underline{v}} \ln f'\right) \cdot \underline{\underline{U}}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} \ln f - \frac{\partial}{\partial \underline{v}'} \ln f'\right)$$

where

$$\underline{\underline{U}}(\underline{u}) = \frac{u^2 \underline{\underline{I}} - \underline{u}\,\underline{u}}{u^3}$$

So

$$\underline{c} \cdot \underline{\underline{U}} \cdot \underline{c} = \frac{u^2 c^2 - (\underline{c} \cdot \underline{u})^2}{u^3}$$

But

$$\frac{|\underline{u} \times \underline{c}|^2}{u^3} = \frac{1}{u^3}(\underline{u} \times \underline{c}) \cdot (\underline{u} \times \underline{c}) = \frac{1}{u^3}\left(\underline{c} \times (\underline{u} \times \underline{c})\right) \cdot \underline{u}$$

$$= \frac{1}{u^3}\left(\underline{u} c^2 - \underline{c}\, \underline{c} \cdot \underline{u}\right) \cdot \underline{u} = \frac{1}{u^3}\left(u^2 c^2 - (\underline{c} \cdot \underline{u})^2\right) = \underline{c} \cdot \underline{\underline{U}} \cdot \underline{c}$$

So we have $\underline{c}\cdot\underline{\underline{U}}\cdot\underline{c} = \frac{|\underline{u}\times\underline{c}|}{u^3} \geq 0$

Here $\underline{c} = \frac{\partial}{\partial \underline{v}}\ln f - \frac{\partial}{\partial \underline{v}'}\ln f'$

Because of its positivity (we'll prove later in Problem 5), $f \geq 0$

$f' \geq 0$.

$$\frac{ds}{dt} = \frac{1}{2}\Gamma\int d^3v\, d^3v'\, f(\underline{v}) f(\underline{v}')\, \frac{|\underline{u}\times\underline{c}|^2}{u^3} \cdot \geq 0.$$

When $\frac{ds}{dt} = 0$, then entropy reaches its maximum.

we require $\underline{u}\times\underline{c} = 0$, i.e. $\underline{u} \parallel \underline{c}$. So

$$\frac{\partial}{\partial \underline{v}}\ln f - \frac{\partial}{\partial \underline{v}'}\ln f = C_0(\underline{v} - \underline{v}')$$

$\Rightarrow \frac{\partial}{\partial \underline{v}}\ln f = C_0(\underline{v} - \underline{V})$, where C_0 & $\underline{V}$ are arbitrary constants.

Integrate above, we obtain

$$f = C_1 e^{-\frac{C_0}{2}(\underline{v} - \underline{V})^2} \qquad C_1 \text{ another constant}$$

Let $\frac{C_0}{2} \equiv \frac{1}{v_T^2}$, $C_1 = \frac{1}{\sqrt{\pi v_T^2}}$, then

$$f = \frac{1}{\sqrt{\pi v_T^2}} e^{-\frac{(v-V)^2}{v_T^2}} \qquad \text{(drift Maxwellian)}$$

5. Positivity of f.

Consider a 1-D case. f satisfy the Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}(Af) + \frac{1}{2}\frac{\partial^2}{\partial v^2}(Df)$$

$$= -\frac{\partial A}{\partial v} f - A\frac{\partial f}{\partial v} + \frac{1}{2}\frac{\partial^2 D}{\partial v^2} f + \frac{\partial D}{\partial v}\frac{\partial f}{\partial v} + \frac{1}{2} D \frac{\partial^2 f}{\partial v^2}$$

At $t=0$, $f>0$, As shown in figure 1

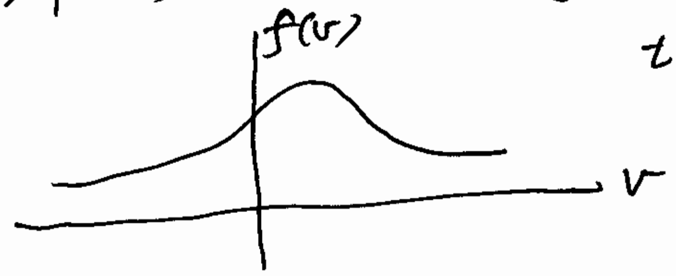


fig. 1.

Then f change continuously according to the Fokker-Planck Equation

Assume some time t_0 later, f reaches to zero for the first time

at $v=v_0$, i.e. $f(v_0,t_0)=0$, see fig 2.

Because of the continuity of f. $f(v_0,t_0)$ must be a

minimum at t_0. i.e.

$$\left.\frac{\partial f}{\partial v}\right|_{v_0,t_0} = 0, \quad \left.\frac{\partial^2 f}{\partial v^2}\right|_{v_0,t_0} > 0$$

Therefore. According to Fokker-Plack Equation

$$\left.\frac{\partial f}{\partial t}\right|_{t_0, v_0} = \frac{1}{2} D \left.\frac{\partial^2 f}{\partial v^2}\right|_{t_0 v_0} > 0.$$

So At a little later time $t_0+\Delta t$

$$f(t_0+\Delta t, v_0) = \frac{1}{2} D \left.\frac{\partial^2 f}{\partial v^2}\right|_{v_0 t_0} \Delta t > 0, \quad \Delta t \to 0.$$

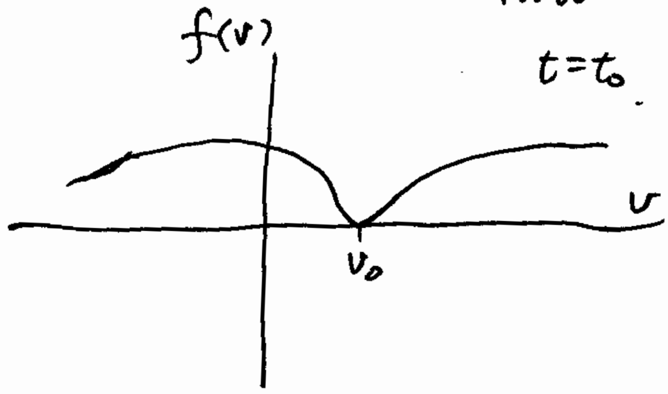


fig. 2

i.e. The diffusion factor try to smooth the ~~dif~~ distribution curve.

f try to keep above ~~0~~ zero.

So ~~we~~ if $f>0$ at $t=0$, f evolves according to Fokker-Planck equation, then $f>0$ for all times $t>0$.