10.5 Center of Mass

Consider two point-like particles with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\vec{r}_1$ and body 2 has position $\vec{r}_2$ (Figure 10.5).
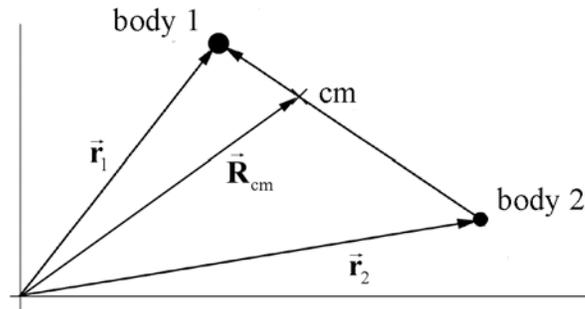

Figure 10.5 Center of mass coordinate system.

The center of mass vector, $\vec{R}_{cm}$, of the two-body system is defined as

$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} .$$
(10.5.1)

We shall now extend the concept of the center of mass to more general systems. Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. Choose a coordinate system and denote the position of the ith particle as $\vec{r}_i$. The mass of the system is given by the sum

$$m_{\text{sys}} = \sum_{i=1}^{i=N} m_i \qquad (10.5.2)$$

and the position of the center of mass of the system of particles is given by

$$\vec{R}_{\text{cm}} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{r}_i . \qquad (10.5.3)$$

(For a continuous rigid body, each point-like particle has mass dm and is located at the position $\vec{r}'$. The center of mass is then defined as an integral over the body,

$$\vec{R}_{\text{cm}} = \frac{\displaystyle\int_{\text{body}} dm\, \vec{r}'}{\displaystyle\int_{\text{body}} dm} . \qquad (10.5.4)$$

Example 10.2 Center of Mass of the Earth-Moon System

The mean distance from the center of the earth to the center of the moon is $r_{em} = 3.84 \times 10^8$ m. The mass of the earth is $m_e = 5.98 \times 10^{24}$ kg and the mass of the moon is $m_m = 7.34 \times 10^{22}$ kg. The mean radius of the earth is $r_e = 6.37 \times 10^6$ m. The mean radius of the moon is $r_m = 1.74 \times 10^6$ m. Where is the location of the center of mass of the earth-moon system? Is it inside the earth's radius or outside?

Solution: The center of mass of the earth-moon system is defined to be

$$\vec{R}_{cm} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{r}_i = \frac{1}{m_e + m_m} (m_e \vec{r}_e + m_m \vec{r}_m) . \qquad (10.5.5)$$

Choose an origin at the center of the earth and a unit vector $\hat{\mathbf{i}}$ pointing towards the moon, then $\vec{r}_e = \vec{0}$. The center of mass of the earth-moon system is then

$$\vec{R}_{cm} = \frac{1}{m_e + m_m}(m_e \vec{r}_e + m_m \vec{r}_m) = \frac{m_m \vec{r}_{em}}{m_e + m_m} = \frac{m_m r_{em}}{m_e + m_m} \hat{\mathbf{i}} \qquad (10.5.6)$$

$$\vec{R}_{cm} = \frac{(7.34 \times 10^{22}\ \text{kg})(3.84 \times 10^8\ \text{m})}{(5.98 \times 10^{24}\ \text{kg} + 7.34 \times 10^{22}\ \text{kg})} \hat{\mathbf{i}} = 4.66 \times 10^6\ \text{m}\ \hat{\mathbf{i}} \qquad (10.5.7)$$

The earth's mean radius is $r_e = 6.37 \times 10^6$ m so the center of mass of the earth-moon system lies within the earth.

Example 10.3 Center of Mass of a Rod

A thin rod has length L and mass M.

$\lambda = M / L$

M

a) uniform rod

$\lambda(x) = \dfrac{\lambda_0}{L^2} x^2$

M
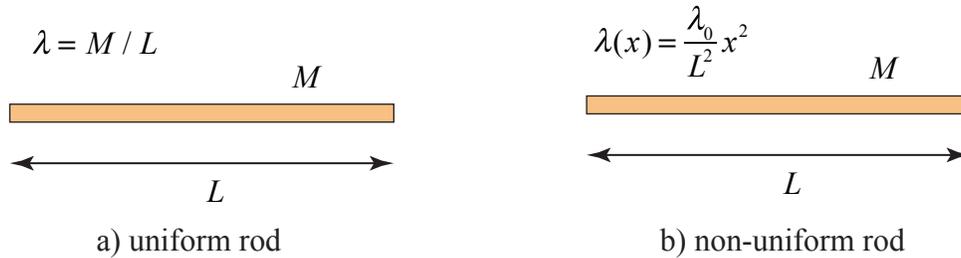
b) non-uniform rod

Figure 10.6 a) Uniform rod and b) non-uniform rod

(a) Suppose the rod is uniform (Figure 10.6a). Find the position of the center of mass with respect to the left end of the rod. (b) Now suppose the rod is not uniform (Figure 10.6b) with a linear mass density that varies with the distance x from the left end according to

$$\lambda(x) = \frac{\lambda_0}{L^2} x^2 \qquad (10.5.8)$$

where λ_0 is a constant and has SI units $[\text{kg} \cdot \text{m}^{-1}]$. Find λ_0 and the position of the center of mass with respect to the left end of the rod.

Solution: (a) Choose a coordinate system with the rod aligned along the x-axis and the origin located at the left end of the rod. The center of mass of the rod can be found using the definition given in Eq. (10.5.4). In that expression dm is an infinitesimal mass element and $\vec{\mathbf{r}}$ is the vector from the origin to the mass element dm (Figure 10.6c).

$\lambda = M / L$

x'

$+x$

$x = 0$

$dm = \lambda dx'$ $x = L$
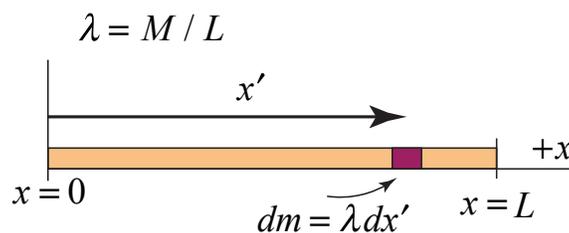
Figure 10.6c Infinitesimal mass element for rod

Choose an infinitesimal mass element dm located a distance x' from the origin. In this problem x' will be the integration variable. Let the length of the mass element be dx'. Then

$$dm = \lambda dx' \qquad (10.5.9)$$

The vector $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}$. The center of mass is found by integration

10-3

$$\vec{R}_{cm} = \frac{1}{M} \int_{body} \vec{r} \, dm = \frac{1}{L} \int_{x'=0}^{x} x' dx' \, \hat{i} = \frac{1}{2L} x'^2 \Big|_{x'=0}^{x'=L} \hat{i} = \frac{1}{2L}(L^2 - 0) \, \hat{i} = \frac{L}{2} \hat{i}. \quad (10.5.10)$$

(b) For a non-uniform rod (Figure 10.6d),
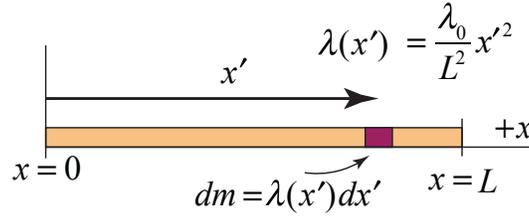
Figure 10.6d Non-uniform rod

the mass element is found using Eq. (10.5.8)

$$dm = \lambda(x')dx' = \lambda = \frac{\lambda_0}{L^2} x'^2 dx' . \quad (10.5.11)$$

The vector $\vec{r} = x' \, \hat{i}$. The mass is found by integrating the mass element over the length of the rod

$$M = \int_{body} dm = \int_{x'=0}^{x=L} \lambda(x')dx' = \frac{\lambda_0}{L^2} \int_{x'=0}^{x=L} x'^2 dx' = \frac{\lambda_0}{3L^2} x'^3 \Big|_{x'=0}^{x'=L} = \frac{\lambda_0}{3L^2}(L^3 - 0) = \frac{\lambda_0}{3} L . \quad (10.5.12)$$

Therefore

$$\lambda_0 = \frac{3M}{L} \quad (10.5.13)$$

The center of mass is again found by integration

$$\vec{R}_{cm} = \frac{1}{M} \int_{body} \vec{r} \, dm = \frac{3}{\lambda_0 L} \int_{x'=0}^{x} \lambda(x')x' \, dx' \, \hat{i} = \frac{3}{L^3} \int_{x'=0}^{x} x'^3 \, dx' \, \hat{i}$$

$$\vec{R}_{cm} = \frac{3}{4L^3} x'^4 \Big|_{x'=0}^{x'=L} \hat{i} = \frac{3}{4L^3}(L^4 - 0) \, \hat{i} = \frac{3}{4} L \, \hat{i}.$$

$$\quad (10.5.14)$$

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8.01 Classical Mechanics
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