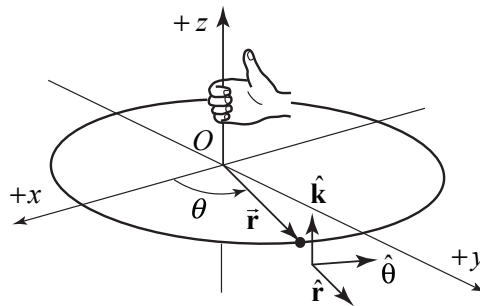


## 6.5 Angular Velocity and Angular Acceleration

### 6.5.1. Angular Velocity

We shall always choose a right-handed cylindrical coordinate system. If the positive  $z$ -axis points up, then we choose  $\theta$  to be increasing in the counterclockwise direction as shown in Figures 6.6.



**Figure 6.6** Right handed coordinate system

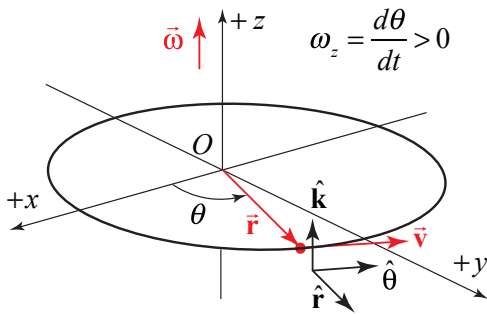
For a point object undergoing circular motion about the  $z$ -axis, the angular velocity vector  $\vec{\omega}$  is directed along the  $z$ -axis with  $z$ -component equal to the time derivative of the angle  $\theta$ ,

$$\vec{\omega} = \frac{d\theta}{dt} \hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}}. \quad (6.5.4)$$

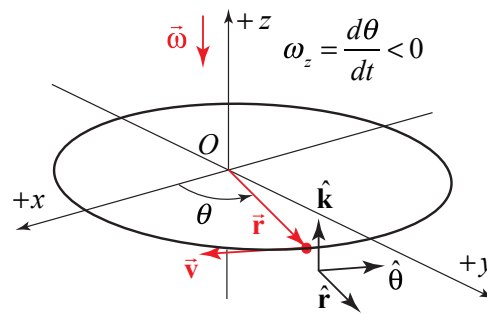
The SI units of angular velocity are  $[\text{rad} \cdot \text{s}^{-1}]$ . Note that the angular speed is just the magnitude of the  $z$ -component of the angular velocity,

$$\omega \equiv |\omega_z| = \left| \frac{d\theta}{dt} \right|. \quad (6.5.5)$$

If the velocity of the object is in the  $+\hat{\theta}$ -direction, (rotating in the counterclockwise direction in Figure 6.7(a)), then the  $z$ -component of the angular velocity is positive,  $\omega_z = d\theta / dt > 0$ . The angular velocity vector then points in the  $+\hat{k}$ -direction as shown in Figure 6.7(a). If the velocity of the object is in the  $-\hat{\theta}$ -direction, (rotating in the clockwise direction in Figure 6.7(b)), then the  $z$ -component of the angular velocity is negative,  $\omega_z = d\theta / dt < 0$ . The angular velocity vector then points in the  $-\hat{k}$ -direction as shown in Figure 6.7(b).



**Figure 6.7(a)** Angular velocity vector for motion with  $d\theta / dt > 0$ .



**Figure 6.7(b)** Angular velocity for motion with  $d\theta / dt < 0$ .

The velocity and angular velocity are related by

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{d\theta}{dt} \hat{k} \times r \hat{r} = r \frac{d\theta}{dt} \hat{\theta}. \quad (6.5.6)$$

### Example 6.2 Angular Velocity

A particle is moving in a circle of radius  $R$ . At  $t = 0$ , it is located on the  $x$ -axis. The angle the particle makes with the positive  $x$ -axis is given by  $\theta(t) = At - Bt^3$ , where  $A$  and  $B$  are positive constants. Determine (a) the angular velocity vector, and (b) the velocity vector. Express your answer in polar coordinates. (c) At what time,  $t = t_1$ , is the angular velocity zero? (d) What is the direction of the angular velocity for (i)  $t < t_1$ , and (ii)  $t > t_1$ ?

**Solution:** The derivative of  $\theta(t) = At - Bt^3$  is

$$\frac{d\theta(t)}{dt} = A - 3Bt^2.$$

Therefore the angular velocity vector is given by

$$\vec{\omega}(t) = \frac{d\theta(t)}{dt} \hat{\mathbf{k}} = (A - 3Bt^2) \hat{\mathbf{k}}.$$

The velocity is given by

$$\vec{v}(t) = R \frac{d\theta(t)}{dt} \hat{\boldsymbol{\theta}}(t) = R(A - 3Bt^2) \hat{\boldsymbol{\theta}}(t).$$

The angular velocity is zero at time  $t = t_1$  when

$$A - 3Bt_1^2 = 0 \Rightarrow t_1 = \sqrt{A/3B}.$$

For  $t < t_1$ ,  $\frac{d\theta(t)}{dt} = A - 3Bt_1^2 > 0$  hence  $\vec{\omega}(t)$  points in the positive  $\hat{\mathbf{k}}$ -direction.

For  $t > t_1$ ,  $\frac{d\theta(t)}{dt} = A - 3Bt_1^2 < 0$  hence  $\vec{\omega}(t)$  points in the negative  $\hat{\mathbf{k}}$ -direction.

### 6.5.2 Angular Acceleration

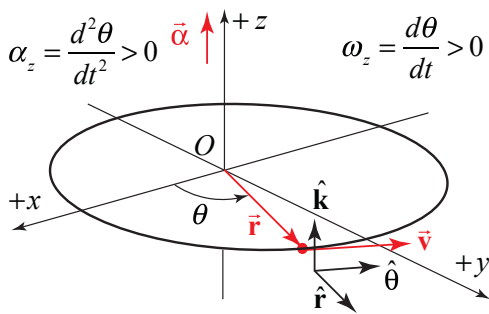
In a similar fashion, for a point object undergoing circular motion about the fixed  $z$ -axis, the angular acceleration is defined as

$$\vec{\alpha} = \frac{d^2\theta}{dt^2} \hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}}. \quad (6.5.7)$$

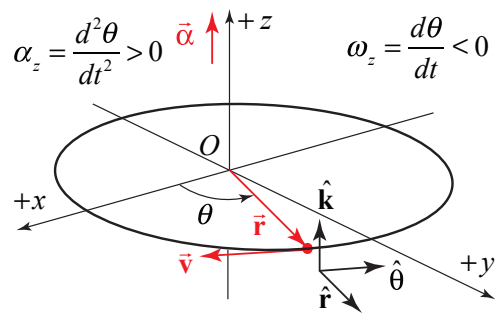
The SI units of angular acceleration are  $[\text{rad} \cdot \text{s}^{-2}]$ . The magnitude of the angular acceleration is denoted by the Greek symbol alpha,

$$\alpha \equiv |\vec{\alpha}| = \left| \frac{d^2\theta}{dt^2} \right|. \quad (6.5.8)$$

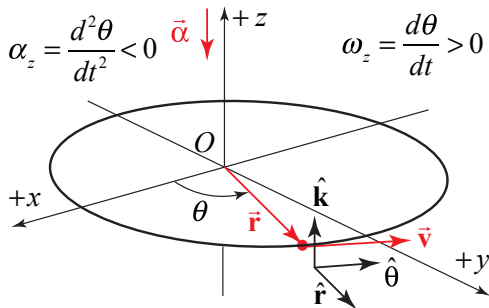
There are four special cases to consider for the direction of the angular velocity. Let's first consider the two types of motion with  $\vec{\alpha}$  pointing in the  $+\hat{k}$ -direction: (i) if the object is rotating counterclockwise and speeding up then both  $d\theta/dt > 0$  and  $d^2\theta/dt^2 > 0$  (Figure 6.8(a)), (ii) if the object is rotating clockwise and slowing down then  $d\theta/dt < 0$  but  $d^2\theta/dt^2 > 0$  (Figure 6.8(b)). There are two corresponding cases in which  $\vec{\alpha}$  pointing in the  $-\hat{k}$ -direction: (iii) if the object is rotating counterclockwise and slowing down then  $d\theta/dt > 0$  but  $d^2\theta/dt^2 < 0$  (Figure 6.9(a)), (iv) if the object is rotating clockwise and speeding up then both  $d\theta/dt < 0$  and  $d^2\theta/dt^2 < 0$  (Figure 6.9(b)).



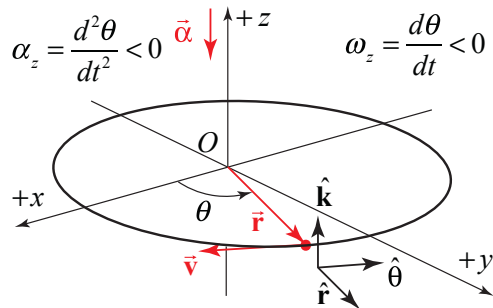
**Figure 6.8(a)** Angular acceleration vector for motion with  $d\theta/dt > 0$ , and  $d^2\theta/dt^2 > 0$ .



**Figure 6.8(b)** Angular velocity for motion with  $d\theta/dt < 0$ , and  $d^2\theta/dt^2 > 0$ .



**Figure 6.9(a)** Angular acceleration vector for motion with  $d\theta/dt > 0$ , and  $d^2\theta/dt^2 < 0$ .



**Figure 6.9(b)** Angular velocity for motion with  $d\theta/dt < 0$ , and  $d^2\theta/dt^2 < 0$ .

### Example 6.3 Integration and Circular Motion Kinematics

A point-like object is constrained to travel in a circle. The  $z$ -component of the angular acceleration of the object for the time interval  $[0, t_1]$  is given by the function

$$\alpha_z(t) = \begin{cases} b \left( 1 - \frac{t}{t_1} \right); & 0 \leq t \leq t_1, \\ 0; & t > t_1 \end{cases},$$

where  $b$  is a positive constant with units  $\text{rad} \cdot \text{s}^{-2}$ .

- Determine an expression for the angular velocity of the object at  $t = t_1$ .
- Through what angle has the object rotated at time  $t = t_1$ ?

#### Solution:

a) The angular velocity at time  $t = t_1$  is given by

$$\omega_z(t_1) - \omega_z(t=0) = \int_{t'=0}^{t'=t_1} \alpha_z(t') dt' = \int_{t'=0}^{t'=t_1} b \left( 1 - \frac{t'}{t_1} \right) dt' = b \left( t_1 - \frac{t_1^2}{2t_1} \right) = \frac{bt_1}{2}$$

b) In order to find the angle  $\theta(t_1) - \theta(t=0)$  that the object has rotated through at time  $t = t_1$ , you first need to find  $\omega_z(t)$  by integrating the  $z$ -component of the angular acceleration

$$\omega_z(t) - \omega_z(t=0) = \int_{t'=0}^{t'=t} \alpha_z(t') dt' = \int_{t'=0}^{t'=t} b \left( 1 - \frac{t'}{t_1} \right) dt' = b \left( t - \frac{t^2}{2t_1} \right).$$

Because it started from rest,  $\omega_z(t=0) = 0$ , hence  $\omega_z(t) = b \left( t - \frac{t^2}{2t_1} \right)$ ;  $0 \leq t \leq t_1$ .

Then integrate  $\omega_z(t)$  between  $t = 0$  and  $t = t_1$  to find that

$$\theta(t_1) - \theta(t=0) = \int_{t'=0}^{t'=t_1} \omega_z(t') dt' = \int_{t'=0}^{t'=t_1} b \left( t' - \frac{t'^2}{2t_1} \right) dt' = b \left( \frac{t_1^2}{2} - \frac{t_1^3}{6t_1} \right) = \frac{bt_1^2}{3}.$$

## 6.5 Non-circular Central Planar Motion

Let's now consider central motion in a plane that is non-circular. In Figure 6.10, we show the spiral motion of a moving particle. In polar coordinates, the key point is that the time derivative  $dr/dt$  of the position function  $r$  is no longer zero. The second derivative  $d^2r/dt^2$  also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is still given by Eq. (6.2.1), which we shall repeat below

$$\vec{\mathbf{r}} = r \hat{\mathbf{r}}. \quad (6.5.9)$$

Because  $dr/dt \neq 0$ , when we differentiate Eq. (6.5.9), we need to use the product rule

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\hat{\mathbf{r}}}{dt}. \quad (6.5.10)$$

Substituting Eq. (6.2.4) into Eq. (6.5.10)

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} = v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}}. \quad (6.5.11)$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}. \quad (6.5.12)$$

In order to determine the acceleration, we now differentiate Eq. (6.5.11), again using the product rule, which is now a little more involved:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2} \hat{\mathbf{r}} + \frac{dr}{dt} \frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt} \frac{d\theta}{dt} \hat{\boldsymbol{\theta}} + r \frac{d^2\theta}{dt^2} \hat{\boldsymbol{\theta}} + r \frac{d\theta}{dt} \frac{d\hat{\boldsymbol{\theta}}}{dt}. \quad (6.5.13)$$

Now substitute Eqs. (6.2.4) and (6.2.7) for the time derivatives of the unit vectors in Eq. (6.5.13), and after collecting terms yields

$$\begin{aligned} \vec{\mathbf{a}} &= \left( \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \hat{\mathbf{r}} + \left( 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \hat{\boldsymbol{\theta}} \\ &= a_r \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}} \end{aligned} \quad (6.5.14)$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of  $dr/dt$  and  $d^2r/dt^2$ . The radial component is

$$a_r = \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2. \quad (6.5.15)$$

and the tangential component is

$$a_\theta = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}. \quad (6.5.16)$$

The first term in the tangential component of the acceleration,  $2(dr/dt)(d\theta/dt)$  has a special name, the *Coriolis acceleration*,

$$a_{cor} = 2 \frac{dr}{dt} \frac{d\theta}{dt}. \quad (6.5.17)$$

### Example 6.4 Spiral Motion

A particle moves outward along a spiral starting from the origin at  $t = 0$ . Its trajectory is given by  $r = b\theta$ , where  $b$  is a positive constant with units  $[m \cdot rad^{-1}]$ .  $\theta$  increases in time according to  $\theta = ct^2$ , where  $c > 0$  is a positive constant (with units  $[rad \cdot s^{-2}]$ ).

- Determine the acceleration as a function of time.
- Determine the time at which the radial acceleration is zero.
- What is the angle when the radial acceleration is zero?
- Determine the time at which the radial and tangential accelerations have equal magnitude.

#### Solution:

a) The position coordinate as a function of time is given by  $r = b\theta = bct^2$ . The acceleration is given by Eq. (6.5.14). In order to calculate the acceleration, we need to calculate the four derivatives  $dr/dt = 2bct$ ,  $d^2r/dt^2 = 2bc$ ,  $d\theta/dt = 2ct$ , and  $d^2\theta/dt^2 = 2c$ . The acceleration is then

$$\bar{\mathbf{a}} = (2bc - 4bc^3t^4) \hat{\mathbf{r}} + (8bc^2t^2 + 2bc^2t^2) \hat{\boldsymbol{\theta}} = (2bc - 4bc^3t^4) \hat{\mathbf{r}} + 10bc^2t^2 \hat{\boldsymbol{\theta}}.$$

b) The radial acceleration is zero when

$$t_1 = \left( \frac{1}{2c^2} \right)^{1/4}.$$

c) The angle when the radial acceleration is zero is

$$\theta_1 = ct_1^2 = \sqrt{2}/2.$$

- d) The radial and tangential accelerations have equal magnitude when after some algebra

$$(2bc - 4bc^3t^4) = 10bc^2t^2 \Rightarrow 0 = t^4 + (5/2c)t^2 - (1/2c^2).$$

This equation has as only positive solution for  $t^2$ :

$$t_2^2 = \frac{-(5/2c) \pm ((5/2c)^2 + 2c^2)^{1/2}}{2} = \frac{\sqrt{33} - 5}{4c}.$$

Therefore the magnitudes of the two components are equal when

$$t_2 = \sqrt{\frac{\sqrt{33} - 5}{4c}}.$$



MIT OpenCourseWare  
<https://ocw.mit.edu>

8.01 Classical Mechanics  
Fall 2016

For Information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.