MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Problem Solving 10: The Displacement Current and Poynting Vector

OBJECTIVES

1. To introduce the “displacement current” term that Maxwell added to Ampere’s Law

2. To find the magnetic field inside a charging cylindrical capacitor using this new term in Ampere’s Law.

3. To introduce the concept of energy flow through space in the electromagnetic field.

4. To quantify that energy flow by introducing the Poynting vector.

5. To do a calculation of the rate at which energy flows into a capacitor when it is charging, and show that it accounts for the rate at which electric energy stored in the capacitor is increasing.

REFERENCE: Sections 13-1 and 13-6, 8.02 Course Notes.

The Displacement Current

In magnetostatics (the electric and magnetic fields do not change with time), Ampere’s law established a relation between the line integral of the magnetic field around a closed path and the current flowing across any open surface with that closed path as a boundary of the open surface,

$$\oint_{\substack{\text{closed}\\\text{path}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} = \mu_0 \iint_{\substack{\text{open}\\\text{surface}}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} \,.$$

For reasons we have discussed in class, Maxwell argued that in time-dependent situations this equation was incomplete and that an additional term should be added:

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \tag{9.1}$$

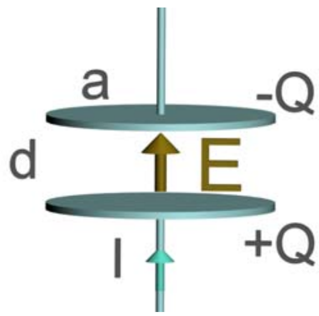
or

$$\oint_{\substack{\text{closed}\\\text{loop}}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} + \mu_0 I_d \tag{9.2}$$

where $I_d = \varepsilon_0 \dfrac{d\Phi_E}{dt}$ is the displacement current.

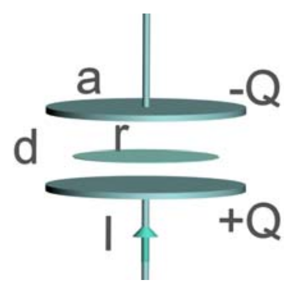
An Example: The Charging Capacitor

A capacitor consists of two circular plates of radius *a* separated by a distance *d* (assume $d << a$). The center of each plate is connected to the terminals of a voltage source by a thin wire. A switch in the circuit is closed at time $t = 0$ and a current $I(t)$ flows in the circuit. The charge on the plate is related to the current according to $I(t) = \frac{dQ(t)}{dt}$. We begin by calculating the electric field between the plates. Throughout this problem you may ignore edge effects. We assume that the electric field is zero for $r > a$.



Question 1: Use Gauss' Law to find the electric field between the plates when the charge on them is *Q* (as pictured). The vertical direction is the $\hat{\mathbf{k}}$ direction.
Answer ***(write your answer to this and subsequent questions on the tear-sheet!)*****:**

Now take an imaginary flat disc of radius $r < a$ inside the capacitor, as shown below.



Question 2: Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through this flat disc of radius $r < a$ in the plane midway between the plates. Take the surface normal to the imaginary disk to be in the $\hat{\mathbf{k}}$ direction.
Answer: $\Phi_E = \iint_{\substack{flat\\disk}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} =$

This electric flux is changing in time because as the plates are charging up, the electric field is increasing with time.

Question 3: Calculate the Maxwell displacement current,

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 \frac{d}{dt} \iint_{disc(r)} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

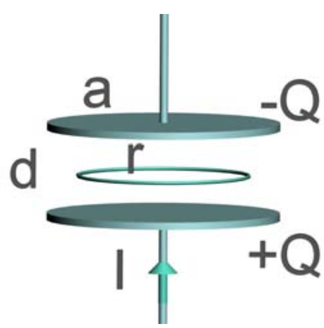
through the flat disc of radius $r < a$ in the plane midway between the plates, in terms of r, $I(t)$, and a.
Answer:

Question 4: What is the conduction current $\iint_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$ through the flat disc of radius $r < a$?

"Conduction" current just means the current due to the flow of real charge across the surface (e.g. electrons or ions).

Answer:

Since the capacitor plates have an axial symmetry and we know that the magnetic field due to a wire runs in azimuthal circles about the wire, we assume that the magnetic field between the plates is non-zero, and also runs in azimuthal circles.



Question 5: Choose for an Amperian loop a circle of radius $r < a$ in the plane midway between the plates. Calculate the line integral of the magnetic field around the circle, $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$. Express your answer in terms of $|\vec{\mathbf{B}}|$ and r. The line element $d\vec{\mathbf{s}}$ is right-handed with respect to $d\vec{\mathbf{A}}$, that is counterclockwise as seen from the top.

Answer: $\oint_{\text{circle}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} =$

Question 6: Now use the results of your answers above, and apply the generalized Ampere' Law Equation (9.1) or (9.2), find the magnitude of the magnetic field at a distance $r < a$ from the axis.

Answer:

Question 7: If you use your right thumb to point along the direction of the electric field, as the plates charge up, does the magnetic field point in the direction your fingers curl on your right hand or opposite the direction your fingers curl on your right hand?

Answer:

Question 8: Would the direction of the magnetic field change if the plates were discharging? Why or why not?

Answer:

The Poynting Vector

Once a capacitor has been charged up, it contains electric energy. We know that the energy stored in the capacitor came from the battery. How does that energy get from the battery to the capacitor? Energy flows through space from the battery into the ***sides*** of the capacitor. In electromagnetism, the rate of energy flow per unit area is given by the Poynting vector

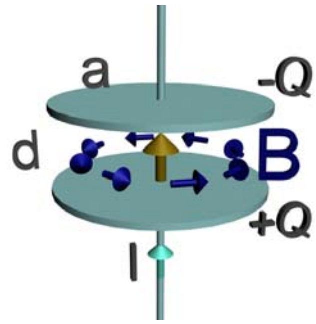
$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}} \quad \text{(units: } \frac{\text{joules}}{\text{sec square meter}}\text{)}$$

To calculate the amount of electromagnetic energy flowing through a surface, we calculate the surface integral $\iint \vec{\mathbf{S}}\cdot d\vec{\mathbf{A}}$ (units: $\frac{\text{joules}}{\text{sec}}$ or watts) .

Energy Flow in a Charging Capacitor

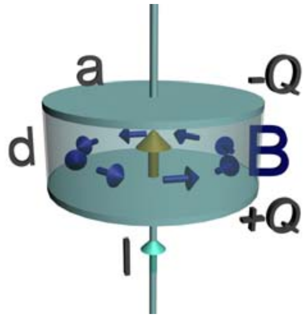
We show how to do a Poynting vector calculation by explicitly calculating the Poynting vector inside a charging capacitor. The electric field and magnetic fields of a charging cylindrical capacitor are (ignoring edge effects)

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q(t)}{\pi a^2 \varepsilon_0}\hat{\mathbf{k}} & r \le a \\ \vec{\mathbf{0}} & r > a \end{cases} \qquad \vec{\mathbf{B}} = \begin{cases} \dfrac{\mu_0 I(t)}{2\pi a}\dfrac{r}{a}\hat{\boldsymbol{\varphi}} & r < a \\ \dfrac{\mu_0 I(t)}{2\pi r}\hat{\boldsymbol{\varphi}} & r > a \end{cases}$$



Question 9: What is the Poynting vector for $r \le a$?

Since the Poynting vector points radially into the capacitor, electromagnetic energy is flowing into the capacitor through the sides. To calculate the total energy flow into the capacitor, we evaluate the Poynting vector right at $r = a$ and integrate over the sides $r = a$.



Question 10: Calculate the flux $\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$ of the Poynting vector evaluated at $r = a$ through an imaginary cylindrical surface of radius a and height d , i.e. over the side of the capacitor. Your answer should involve *Q, a, I,* and *d.* What are the units of this expression?

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Question 11: The capacitance of a parallel plate capacitor is $C = \frac{\varepsilon_o Area}{d} = \frac{\varepsilon_o \pi a^2}{d}$.
Rewrite your answer to Question 2 above using the capacitance *C*. Your answer should involve only *Q, I,* and *C*.

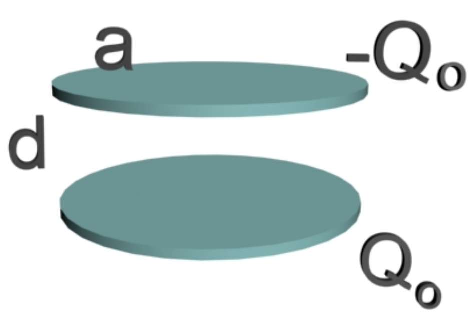
Question 12: The total electrostatic energy stored in the capacitor at time t is given by $\frac{1}{2}\frac{Q(t)^2}{C}$. Show that the rate at which this energy is increasing as the capacitor is charged is equal to the rate at which energy is flowing into the capacitor through the sides, as calculated in Question 3 above. That is, where this energy is coming from is from the flow of energy through the sides of the capacitor.

Question 13: Suppose the capacitor is discharging instead of charging, i.e. $Q(t) > 0$ but now $\frac{dQ(t)}{dt} > 0$ What changes in the picture above? Explain.

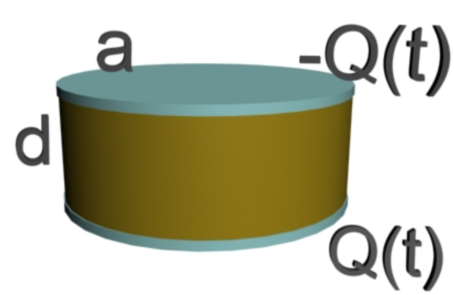
Sample Exam Question (If time, try to do this by yourself, closed notes)

A capacitor consists of two parallel circular plates of radius *a* separated by a distance *d* (assume $a >> d$). The capacitor is initially charged to a charge Q_o. At *t = 0*, this capacitor begins to discharge because we insert a circular resistor of radius *a* and height *d* between the plates, such that the ends of the resistor make good electrical contact with the plates of the capacitor. The capacitor then discharges through this resistor for $t \geq 0$, so the charge on the capacitor becomes a function of time *Q(t)*. *Throughout this problem you may ignore edge effects.*

$t < 0$

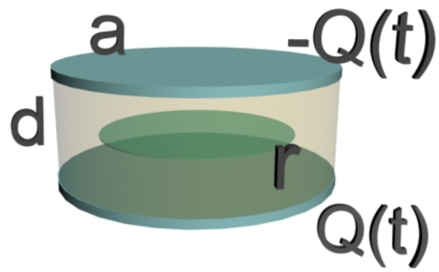


$t \geq 0$



a). Use Gauss's Law to find the electric field between the plates. Is this electric field upward or downward?

b). For $t \geq 0$, consider an imaginary open surface of radius r < a inside the capacitor with its normal $\mathbf{d\vec{A}}$ upward (see figure)



For $t \geq 0$, what is the current flowing through this open surface in terms of *Q(t)* or $\frac{dQ(t)}{dt}$ and the parameters given. Define the direction of positive current to be upward, and be careful about signs.

c) For this same imaginary open surface, what is the time rate of change of the electric flux though the surface, in terms of *Q(t)* or $\frac{dQ(t)}{dt}$ and the parameters given (hint: use your answer above for **E**).

d) What is the integral of the magnetic field around the contour bounding this open circle, using the Ampere-Maxwell Law? Be careful of signs.

e) Does your answer in (d) make sense in terms of the energy dissipation and energy flow in this problem? You must explain your answer clearly and logically to get credit.

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Tear off this page and turn it in at the end of class !!!!

Note:
Writing in the name of a student who is not present is a Committee on Discipline offense.

Problem Solving 11: Displacement Current and Poynting Vector

Group __________________________________ **(e.g. 6A Please Fill Out)**

Names __________________________________

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Question 1: Use Gauss' Law to find the electric field between the plates as a function of time t, in terms of $Q(t)$, a, ε_0, and π.

Answer:

Question 2: Using your expression for $\vec{\mathbf{E}}$ above, calculate the electric flux through the flat disc of radius $r < a$.

Answer:

Question 3: Calculate the Maxwell displacement current $I_d = \varepsilon_0 \dfrac{d\Phi_E}{dt}$ through the disc.

Answer:

Question 4: What is the conduction current through the flat disc of radius $r < a$?

Answer:

Question 5: Calculate the line integral of the magnetic field around the circle, $\oint_{circle} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$.

Answer:

Question 6: What is the magnitude of the magnetic field at a distance $r < a$ from the axis. Your answer should be in terms of *r, I(t),* μ_o, π, and *a.*
Answer:

Question 7: If you use your right thumb to point along the direction of the electric field, as the plates charge up, does the magnetic field point in the direction your fingers curl on your right hand or opposite the direction your fingers curl on your right hand?
Answer:

Question 8: Would the direction of the magnetic field change if the plates were discharging? Why or why not?
Answer:

Question 9: What is the Poynting vector for $r \le a$?

Answer:

Question 10: Calculate the flux $\iint \vec{\mathbf{S}} \cdot d\vec{\mathbf{A}}$ of the Poynting vector evaluated at *r* = *a* through an imaginary cylindrical surface of radius *a* and height *d*, with area $A = 2\pi ab$, i.e. over the sides of the capacitor. Your answer should involve *Q, a, I, d,* π, and ε_o. What are the units of this expression?
Answer:

Question 11: Rewrite your answer to Question 2 above using the capacitance *C*.
Answer:

Question 12: The total electrostatic energy stored in the capacitor at time *t* is given by $Q^2(t)/2C$. Show that the rate at which this energy is increasing as the capacitor is charged is equal to the rate at which energy is flowing into the capacitor through the sides, as calculated in Question 3 above. That is, where this energy is coming from is from the flow of energy through the sides of the capacitor.
Answer:

Question 13: Suppose the capacitor is discharging instead of charging, i.e. $Q(t) > 0$ but now $dQ(t)/dt < 0$ What changes in the picture above? Explain.
Answer: