8.022 (E&M) – Lecture 4

Topics:

- More applications of vector calculus to electrostatics:
  - Laplacian: Poisson and Laplace equation
  - Curl: concept and applications to electrostatics
- Introduction to conductors

Last time…

- Electric potential: $\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{s} \quad \text{with} \quad \vec{E} = -\nabla\phi$
  - Work done to move a unit charge from infinity to the point P(x,y,z)
  - It's a scalar!
- Energy associated with an electric field:
  - Work done to assemble system of charges is stored in E

$$U = \frac{1}{2} \int_{\substack{\text{Volume} \\ \text{with} \\ \text{charges}}} \rho\phi(r)dV = \int_{\substack{\text{Entire} \\ \text{space}}} \frac{E^2}{8\pi} dV$$

- Gauss's law in differential form: $\nabla \cdot \vec{E} = 4\pi\rho$
  - Easy way to go from E to charge distribution that created it

Laplacian operator

What if we combine gradient and divergence?

Let's calculate the div grad f (Q: difference wrt grad div f ?)

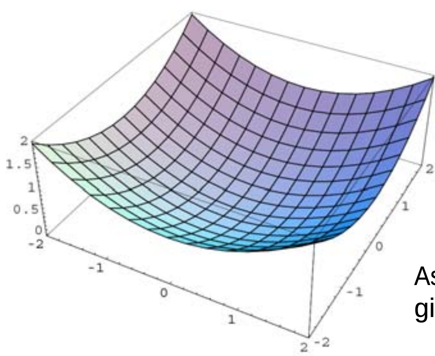
$$\nabla \bullet \nabla f = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right) \bullet \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \equiv \nabla^2 f$$

$$\boxed{\nabla^2 f \equiv \nabla \bullet \nabla f}$$ Laplacian Operator

Interpretation of Laplacian

Given a 2d function $\phi(x,y)=a(x^2+y^2)/4$ calculate the Laplacian



$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f =$$

$$= \frac{a}{4}(2+2) = a$$

As the second derivative, the Laplacian gives the curvature of the function

Poisson equation

Let's apply the concept of Laplacian to electrostatics.

- Rewrite Gauss's law in terms of the potential

$$\begin{cases} \nabla \bullet \vec{E} = 4\pi\rho \\ \nabla \bullet \vec{E} = \nabla \bullet (-\nabla \phi) = -\nabla^2 \phi \end{cases}$$

$$\rightarrow \ \nabla^2 \phi = -4\pi\rho \quad \text{Poisson Equation}$$

Laplace equation and Earnshaw's Theorem

- What happens to Poisson's equation in vacuum?

$$\nabla^2 \phi = -4\pi\rho \ \Rightarrow \ \boxed{\nabla^2 \phi = 0 \quad \text{Laplace Equation}}$$

- What does this teach us?

  In a region where ϕ satisfies Laplace's equation, then its curvature must be 0 everywhere in the region

  → The potential has no local maxima or minima in that region

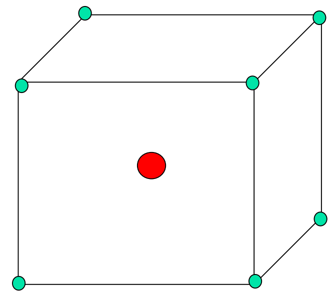
- Important consequence for physics:

  Earnshaw's Theorem:

  It is impossible to hold a charge in stable equilibrium with electrostatic fields (no minima)

Application of Earnshaw's Theorem

8 charges on a cube and one free in the middle.
Is the equilibrium stable? No!

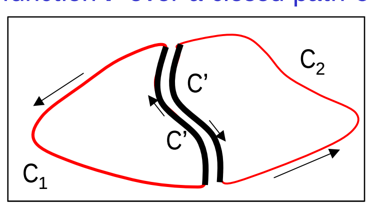


(does the question sound familiar?)

The circulation

- Consider the line integral of a vector function **F** over a closed path C:

$$\Gamma = \oint_C \vec{F} \bullet d\vec{s} \quad \text{Circulation}$$



- Let's now cut C into 2 smaller loops: C_1 and C_2
- Let's write the circulation C in terms of the integral on C_1 and C_2

$$\Gamma = \oint_C \vec{F} \bullet d\vec{s} = \oint_{C_1 - C'} \vec{F} \bullet d\vec{s} + \oint_{C_2 - C'} \vec{F} \bullet d\vec{s} =$$

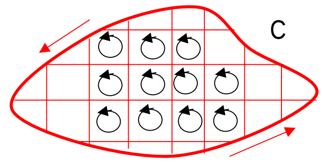
$$\oint_{C_1} \vec{F} \bullet d\vec{s} - \oint_{C'} \vec{F} \bullet d\vec{s} + \oint_{C_2} \vec{F} \bullet d\vec{s} + \oint_{C'} \vec{F} \bullet d\vec{s}$$

$$= \oint_{C_1} \vec{F} \bullet d\vec{s} + \oint_{C_2} \vec{F} \bullet d\vec{s} \qquad \Rightarrow \boxed{\Gamma = \Gamma_1 + \Gamma_2}$$

The curl of F

- If we repeat the procedure N times:

$$\Gamma = \sum_{i=1}^{i=\text{LargeN}} \Gamma_i$$



- Define the <u>curl of **F**</u> as circulation of **F** per unit area in the limit A→0

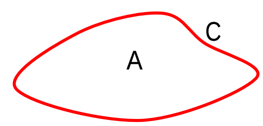
$$\text{curl } \vec{F} \bullet \hat{n} \equiv \lim_{A\to 0} \frac{\oint_C \vec{F} \bullet d\vec{s}}{A}$$

where A is the area inside C

- The curl is a vector normal to the surface A with direction given by the "right hand rule"

Stokes Theorem

$$\Gamma = \sum_{i=1}^{i=\text{LargeN}} \Gamma_i = \sum_{i=1}^{\text{LargeN}} \oint_{C_i} \vec{F} \bullet d\vec{s} = \sum_{i=1}^{\text{LargeN}} A_i \frac{\oint_{C_i} \vec{F} \bullet d\vec{s}}{A_i}$$



$$\text{In the limit } A \to 0\text{:} \quad \frac{\oint_{C_i} \vec{F} \bullet d\vec{s}}{A_i} \to \text{curl } \vec{F} \bullet \hat{n} \quad \text{and} \quad \sum_{i=1}^{i=\text{LargeN}} A_i \to \int_A dA$$

$$\begin{cases} \Gamma = \sum_{i=1}^{\text{LargeN}} A_i \text{ curl } \vec{F} \bullet \hat{n} = \sum_{i=1}^{\text{LargeN}} \text{curl } \vec{F} \bullet (A_i \hat{n}) = \sum_{i=1}^{\text{LargeN}} \text{curl } \vec{F} \bullet \vec{A}_i \to \int_A \text{curl } \vec{F} \bullet d\vec{A} \\ \Gamma = \oint_C \vec{F} \bullet d\vec{s} \qquad \text{(definition of circulation)} \end{cases}$$

$$\Rightarrow \boxed{\oint_C \vec{F} \bullet d\vec{s} = \int_A \text{curl } \vec{F} \bullet d\vec{A}} \quad \text{Stokes Theorem}$$

NB: Stokes relates the <u>line integral</u> of a function **F** over a closed line C and the <u>surface integral of the curl</u> of the function over the area enclosed by C

Application of Stoke's Theorem

- Stoke's theorem:

$$\oint_C \vec{F} \bullet d\vec{s} = \int_A \text{curl } \vec{F} \bullet d\vec{A}$$

- The Electrostatics Force is conservative:

$$\oint_C \vec{F} \bullet d\vec{s} = 0$$

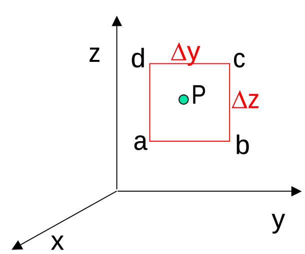
$$\Rightarrow \quad \int_A \text{curl } \vec{E} \bullet d\vec{A} = 0 \ \text{ for any surface A}$$

$$\Rightarrow \quad \boxed{\text{curl } \vec{E} = 0}$$

- The curl of an electrostatic field is zero.

Curl in cartesian coordinates (1)

- Consider infinitesimal rectangle in yz plane centered at P=(x,y,z) in a vector filed **F**
- Calculate circulation of **F** around the square:



$$\begin{cases} \int_a^b \vec{F} \bullet d\vec{s} = F_y(x,y,z-\frac{\Delta z}{2})\Delta y \sim \left[F_y(x,y,z) - \frac{\Delta z}{2}\frac{\partial F_y}{\partial z} \right]\Delta y \\ \int_b^c \vec{F} \bullet d\vec{s} = F_z(x,y+\frac{\Delta y}{2},z)\Delta z \sim \left[F_z(x,y,z) + \frac{\Delta y}{2}\frac{\partial F_z}{\partial y} \right]\Delta z \\ \int_c^d \vec{F} \bullet d\vec{s} = F_y(x,y,z+\frac{\Delta z}{2})(-\Delta y) \sim -\left[F_y(x,y,z) + \frac{\Delta z}{2}\frac{\partial F_y}{\partial z} \right]\Delta y \\ \int_d^a \vec{F} \bullet d\vec{s} = F_z(x,y-\frac{\Delta y}{2},z)(-\Delta z) \sim -\left[F_z(x,y,z) - \frac{\Delta y}{2}\frac{\partial F_z}{\partial y} \right]\Delta z \end{cases}$$

Adding the 4 components: $\Rightarrow$ $$\boxed{\oint_{\text{squareYZ}} \vec{F} \bullet d\vec{s} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right)\Delta y \Delta z}$$

Curl in cartesian coordinates (2)

- Combining this result with definition of curl:

$$\begin{cases} \text{curl } \vec{F} \bullet \hat{n} \equiv \lim_{A \to 0} \dfrac{\oint_C \vec{F} \bullet d\vec{s}}{A} \\ \oint_{\text{square}} \vec{F} \bullet d\vec{s} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} \right) \Delta y \Delta z \end{cases} \Rightarrow \boxed{(\text{curl } \vec{F})_x = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\oint_{square} \vec{F} \bullet d\vec{s}}{\Delta x \Delta y} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} \right)}$$

- Similar results orienting the rectangles in // (xz) and (xy) planes →

$$\text{curl } \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} \equiv \boxed{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \equiv \nabla \times \vec{F}}$$

This is the usable expression for the curl: easy to calculate!

Summary of vector calculus in electrostatics (1)

- Gradient: $\nabla \phi \equiv \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right) \phi$
  - In E&M: $\vec{E} = -\nabla \phi$
- Divergence: $\nabla \bullet \vec{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$
  - Gauss's theorem: $\int_S \vec{E} \bullet d\vec{A} = \int_V \nabla \bullet \vec{E} dV$
  - In E&M: Gauss' law in differential form $\nabla \bullet \vec{E} = 4\pi\rho$
- Curl: $\text{curl } \vec{F} = \nabla \times \vec{F}$
  - Stoke's theorem: $\oint_C \vec{F} \bullet d\vec{s} = \int_A \text{curl } \vec{F} \bullet d\vec{A}$
  - In E&M: $\nabla \times \vec{E} = 0$

Purcell Chapter 2

Summary of vector calculus in electrostatics (2)

- Laplacian: $\nabla^2\phi \equiv \nabla \bullet \nabla\,\phi$
  - In E&M:
    - Poisson Equation: $\nabla^2\phi = -4\pi\rho$
    - Laplace Equation: $\nabla^2\phi = 0$
    - Earnshaw's theorem: impossible to hold a charge in stable equilibrium with electrostatic fields (no local minima)

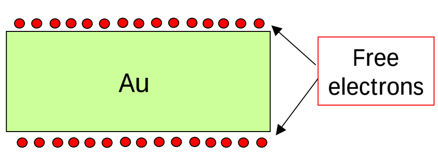
Comment:
This may look like a lot of math: it is!
Time and exercise will help you to learn how to use it in E&M

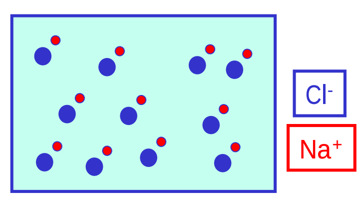
Purcell Chapter 2

Conductors and Insulators

Conductor: a material with free electrons

- Excellent conductors: metals such as Au, Ag, Cu, Al,...
- OK conductors: ionic solutions such as NaCl in H_2O



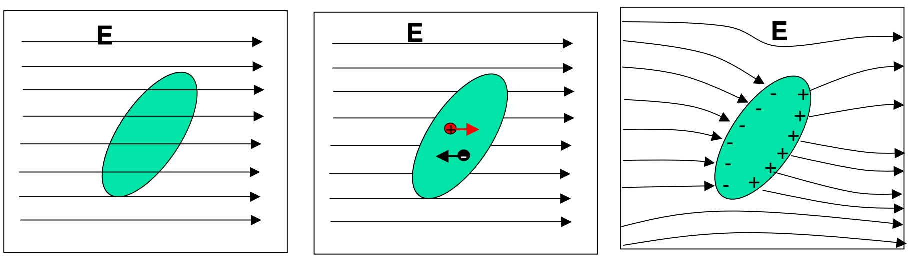


Insulator: a material without free electrons

- Organic materials: rubber, plastic,...
- Inorganic materials: quartz, glass,...

Electric Fields in Conductors (1)

- A conductor is assumed to have an infinite supply of electric charges
  - Pretty good assumption…
- Inside a conductor, **E**=0
  - Why? If **E** is not 0 → charges will move from where the potential is higher to where the potential is lower; migration will stop only when E=0.
  - How long does it take? 10^{-17}-10^{-16} s (typical resistivity of metals)



Electric Fields in Conductors (2)

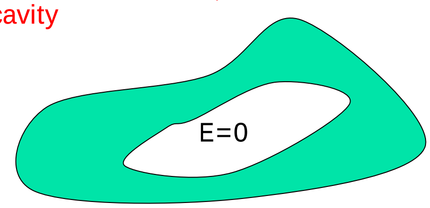
- Electric potential inside a conductor is constant
  - Given 2 points inside the conductor P_1 and P_2 the $\Delta\phi$ would be:

$$\Delta\phi = \int_{P1}^{P2} \vec{E} \bullet d\vec{s} = 0 \quad \text{since E=0 inside the conductor.}$$

- Net charge can only reside on the surface
  - If net charge inside the conductor → Electric Field .ne.0 (Gauss's law)
- External field lines are perpendicular to surface
  - E// component would cause charge flow on the surface until $\Delta\phi$=0
- Conductor's surface is an equipotential
  - Because it's perpendicular to field lines

Corollary 1

In a hollow region inside conductor, ϕ=const and E=0 if there aren't any charges in the cavity



Why?

- Surface of conductor is equipotential
- If no charge inside the cavity → Laplace holds → ϕ_{cavity} cannot have max or minima

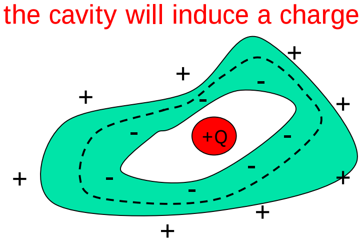
→ ϕ must be constant → E=0

Consequence:

- Shielding of external electric fields: Faraday's cage

Corollary 2

A charge +Q in the cavity will induce a charge +Q on the outside of the conductor



Why?

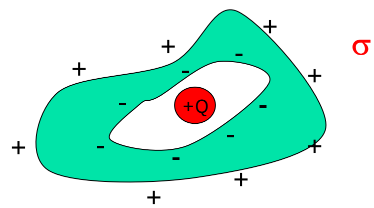
- Apply Gauss's law to surface - - - inside the conductor

$$\begin{cases} \oint \vec{E} \bullet d\vec{A} = 0 \text{ because E=0 inside a conductor} \\ \oint \vec{E} \bullet d\vec{A} = 4\pi(Q + Q_{inside}) \text{ Gauss's law} \end{cases}$$

$$\Rightarrow Q_{inside} = -Q \quad \Rightarrow Q_{outside} = -Q_{inside} = Q \text{ (conductor is overall neutral)}$$

Corollary 3

The induced charge density on the surface of a conductor caused by a charge Q inside it is $\sigma_{induced}=E_{surface}/4\pi$



Why?

- For surface charge layer, Gauss tells us that $\Delta E=4\pi\sigma$
- Since $E_{inside}=0 \rightarrow E_{surface}=4\pi\sigma_{induced}$

Uniqueness theorem

Given the charge density $\rho(x,y,z)$ in a region and the value of the electrostatic potential $\phi(x,y,z)$ on the boundaries, there is only one function $\phi(x,y,z)$ which describes the potential in that region.

Prove:

- Assume there are 2 solutions: ϕ_1 and ϕ_2; they will satisfy Poisson:

$$\nabla^2\phi_1(\vec{r}) = 4\pi\rho(\vec{r})$$
$$\nabla^2\phi_2(\vec{r}) = 4\pi\rho(\vec{r})$$

- Both ϕ_1 and ϕ_2 satisfy boundary conditions: on the boundary, $\phi_1= \phi_2=\phi$
- Superposition: any combination of ϕ_1 and ϕ_2 will be solution, including $\phi_3= \phi_2 - \phi_1$: $\nabla^2\phi_3(\vec{r}) = \nabla^2\phi_2(\vec{r}) - \nabla^2\phi_1(\vec{r}) = 4\pi\rho(\vec{r}) - 4\pi\rho(\vec{r}) = 0$
- ϕ_3 satisfies Laplace: no local maxima or minima inside the boundaries
- On the boundaries $\phi_3=0 \rightarrow \phi_3=0$ everywhere inside region

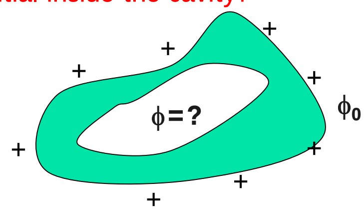
$\rightarrow \phi_1= \phi_2$ everywhere inside region

Why do I care?
A solution is THE solution!

Uniqueness theorem: application 1

- A hollow conductor is charged until its external surface reaches a potential (relative to infinity) $\phi=\phi_0$.

  What is the potential inside the cavity?



Solution

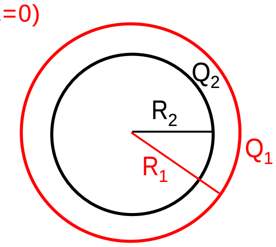
$\phi=\phi_0$ everywhere inside the conductor's surface, including the cavity.

Why? $\phi=\phi_0$ satisfies boundary conditions and Laplace equation

→ The uniqueness theorem tells me that is THE solution.

Uniqueness theorem: application 2

- Two concentric thin conductive spherical shells or radii R1 and R2 carry charges Q_1 and Q_2 respectively.
  - What is the potential of the outer sphere? ($\phi_{infinity}=0$)
  - What is the potential on the inner sphere?
  - What at r=0?



Solution

- Outer sphere: $\phi_1=(Q_1+Q_2)/R_1$
- Inner sphere $\phi_2-\phi_1=-\int_{R2}^{R1}\vec{E}\bullet d\vec{s}=-\int_{R2}^{R1}\frac{Q_2}{r^2}dr=\frac{Q_2}{R_1}-\frac{Q_2}{R_2}$

$$\Rightarrow \phi_2=\frac{Q_2}{R_2}+\frac{Q_1}{R_1} \quad \text{Because of uniqueness: } \phi(r)=\phi_2 \forall r<R_2$$

Next time…

- More on Conductors in Electrostatics
- Capacitors

- NB: All these topics are included in Quiz 1
  scheduled for Tue October 5: just 2 weeks from now!!!

- Reminders:
  - Lab 1 is scheduled for Tomorrow 5-8 pm
  - Pset 2 is due THIS Fri Sep 24