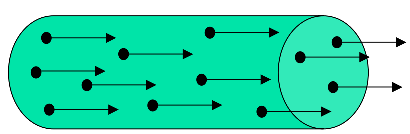
8.022 (E&M) – Lecture 7

Topics:

- Electrical currents
- Conductivity and resistivity
- Ohm's law in microscopic and macroscopic form

Electric current I

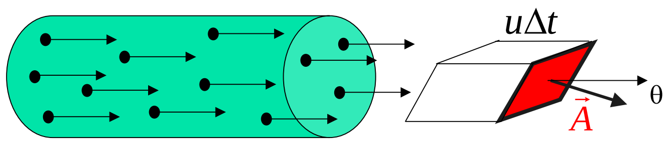
- Consider a region in which there is a flow of charges:
  - E.g. cylindrical conductor



- We define a current:
  the charge/unit time flowing through a certain surface $I = \frac{dQ}{dt}$
- Units:
  - cgs: esu/s
  - SI: C/s=ampere (A)
  - Conversion: 1 A = 2.998 x 10^9 esu/s

Current density J

- Number density: n = #charges / unit volume
- Velocity of each charge: **u**



- Current flowing through area A: I = ΔQ / Δt
  - where ΔQ= q x number of charges in the prism

$$\rightarrow \quad I = \frac{\Delta Q}{\Delta t} = \frac{q\Delta N}{\Delta t} = \frac{qnV_{prism}}{\Delta t} = \frac{qnA\cos\theta u\Delta t}{\Delta t} = qn\vec{u}\bullet\vec{A} = \vec{J}\bullet\vec{A}$$

- Where we defined the current density J as: $\vec{J} \equiv qn\vec{u} \equiv \rho\vec{u}$

NB: ρ=volume charge density

More realistic case...

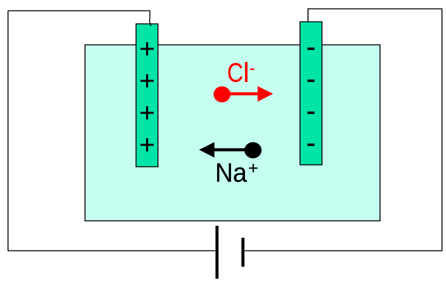
- We made a number of unrealistic assumptions:
  - only 1 kind of charge carriers: we could have several, e.g.: + and – ions
  - **u** assumed to be the same for all particles: unrealistic!
  - regular surface with J constant on it
- Multiple charge carriers: $\vec{J} \equiv \sum_k q_k n_k \vec{u}_k \equiv \sum_k \rho_k \vec{u}_k$
  - E.g.: solution with different kind of ions
  - NB: + ion with velocity u_k is equivalent to – ion with velocity $-u_k$
- Velocity:
  - Not all charges have the same velocity → average velocity $\langle \vec{u}_k \rangle = \frac{1}{N_k}\sum_i (\vec{u}_k)_i$

$$\vec{J} \equiv \sum_k q_k n_k \langle \vec{u}_k \rangle \equiv \sum_k \rho_k \langle \vec{u}_k \rangle$$

- Arbitrary surface S, arbitrary **J**: $I = \int_S \vec{J}\bullet d\vec{A}$

Non standard currents

- We usually think of currents as electrons moving inside a conductor
  - This is only one of the many examples!
- Other kinds of currents
  - Ions in solution such as Salt (NaCl) in water (Demo F5)

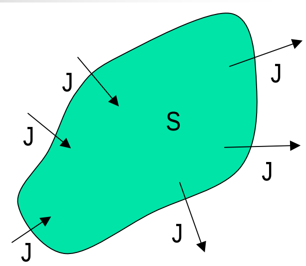


The continuity equation

- A current I flows through the closed surface S:
  - Some charge enters
  - Some charge exits
- What happens to the charge after it enters?
  - Piles up inside
  - Leaves the surface

$$\oint_S \vec{J} \bullet d\vec{A} = -\frac{\partial Q_{inside}}{\partial t}$$

NB: - because dA points outside the surface



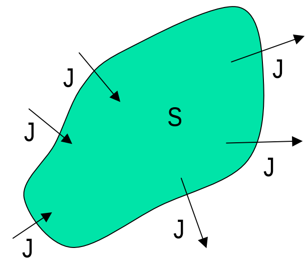
- Apply Gauss's theorem and obtain continuity equation:

$$\begin{cases} \oint_S \vec{J} \bullet d\vec{A} = \oint_V \nabla \bullet \vec{J} dV \\ -\frac{\partial}{\partial t} Q_{inside} = -\frac{\partial}{\partial t} \int_V \rho dV \end{cases} \Rightarrow \int_V \left(\nabla \bullet \vec{J} + \frac{\partial}{\partial t} \rho \right) dV = 0 \Rightarrow \boxed{\nabla \bullet \vec{J} + \frac{\partial}{\partial t} \rho = 0}$$

Thoughts on continuity equation

- Continuity equation:

$$\nabla \bullet \vec{J} + \frac{\partial}{\partial t}\rho = 0$$



- What does it teach us?
  - Conservation of electric charges in presence of currents
  - For steady currents:
    - no accumulation of charges inside the surface: $d\rho/dt=0$

$$\rightarrow \quad \boxed{\nabla \bullet \vec{J} = 0}$$

Microscopic Ohm's law

- Electric fields cause charges to move
- Experimentally, it was observed by Ohm that

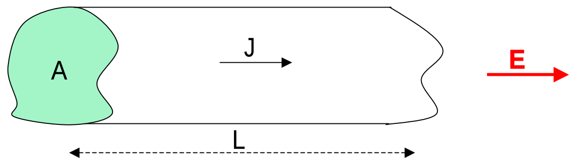
$$\vec{J} = \sigma \vec{E}$$

  - Microscopic version of Ohm's law:
    - It reflects the proportionality between E and J in each point
  - Proportionality constant: conductivity σ

More stuff here pleas

Macroscopic Ohm's law

- Current is flowing in a <u>uniform material</u> of <u>length L</u> in uniform electric field **E** // L



  - Potential difference between two ends: V=EL
  - Ohm's law J=σE holds in every point:

$$J = \sigma E \Rightarrow \frac{I}{A} = \sigma \frac{V}{L} \quad \Rightarrow \quad \boxed{V = IR} \quad \text{where} \quad \boxed{R \equiv \frac{L}{\sigma A}}$$

Resistance R

- Proportionality constant between V and R in Ohm's law

$$R \equiv \frac{L}{\sigma A} \equiv \frac{\rho L}{A}$$

- Units: [V]=[R][I]
  - SI: Ohm (Ω) = V/A
  - cgs: s/cm
- Dependence on the geometry:
  - Inversely proportional to A and proportional to L
- Dependence on the property of the material:
  - Inversely proportional to conductivity

Resistivity

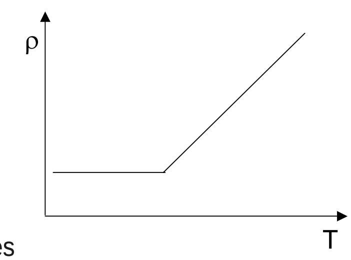
- Resistivity $\rho = 1/\sigma$
  - Describes how fast electrons can travel in the material
  - Units: in SI: Ω m; in cgs: s

Material	Resistivity (Ω-m)	Resistivity (sec)
Silver	1.6×10^{-8}	1.8×10^{-17}
Copper	1.7×10^{-8}	1.9×10^{-17}
Gold	2.4×10^{-8}	2.6×10^{-17}
Iron	1.0×10^{-7}	1.1×10^{-16}
Sea water	0.2	2.2×10^{-10}
Polyethylene	2.0×10^{11}	220
Glass	$\sim 10^{12}$	$\sim 10^{3}$
Fused quartz	7.5×10^{17}	8.3×10^{8}

  - Depends on chemistry of material, temperature,...
    - Demos F1 and F4

Resistivity vs. Temperature

- Does resistivity depend on T?
  - Demos F1 and F4

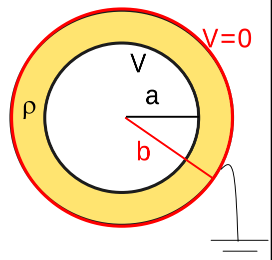


- Why?
  - Room temperature:
    - ρ depends upon collisional processes
      → when T increases → more collisions → ρ increases
  - Very low temperature:
    - Mean free path dominated by impurities or defects in the material → ~ constant with temperature.
    - With sufficient purity, some metals become superconductors

Application:

Resistance of a spherical shell

- 2 concentric spheres; material in between has resistivity ρ
- Difference in potential V → current
  - ϕ_{inner}=V; ϕ_{outer}=0



Q: what is the resistance R?

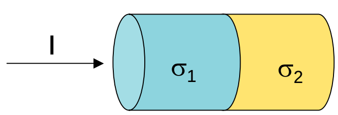
- Microscopic Ohm will hold: $J=\sigma E$
- Spherical symmetry → spherical potential: $\phi(r) = A + \frac{B}{r}$
- Boundary conditions: $\phi(a)$=V and $\phi(b)$=0

$$\rightarrow \quad \phi(r) = V\left(\frac{ab}{b-a}\frac{1}{r} - \frac{a}{b-a}\right)$$

- E=-grad(ϕ): $\vec{E}(r) = V\frac{ab}{b-a}\frac{1}{r^2}\hat{r} \Rightarrow J = \sigma V \frac{ab}{b-a}\frac{1}{r^2}$
- $I = \int_{Sphere} \vec{J}\bullet d\vec{A} = \vec{J}\bullet\vec{A} = 4\pi\sigma V\frac{ab}{b-a} \Rightarrow R = \frac{V}{I} = \frac{V}{4\pi\sigma V\frac{ab}{b-a}} = \boxed{\frac{b-a}{4\pi\sigma ab}}$

What if σ is not constant?

- Cylindrical wire made of 2 conductors with conductivity σ_1 and σ_2



- What is the consequence?
  - Current flowing must be the same in the whole cylinder

$$I = A\sigma_1 E_1 = A\sigma_2 E_2$$

→ Electric fields are different in the 2 regions

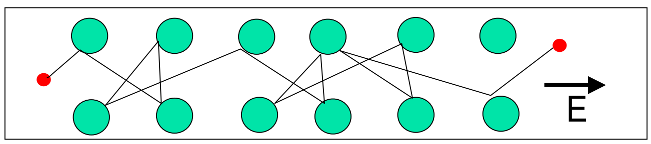
→ E discontinuous → surface layer σ_q at the boundary

$$\sigma_q = \frac{E_{surface}}{4\pi} = \frac{E_2 - E_1}{4\pi} = \frac{I(\rho_2 - \rho_1)}{4\pi A}$$

When conductivity changes there is the possibility that some charge accumulates somewhere. This is necessary to maintain steady flow.

Thoughts on Ohm's law

- Ohm's law in microscopic formulation: $\vec{J} = \sigma \vec{E}$
  - In plain English:
    - A constant electric field creates a steady current: $\vec{E} \propto \vec{v}$
    - Does this make sense? $\vec{F} = m\vec{a} \Rightarrow \vec{E} \propto \vec{a}$
- Charges are moving in an effectively viscous medium
  - As sky diver in free fall: first accelerate, then reach constant v
  - Why? Charges are accelerated by E but then bump into nuclei and are scattered → the average behavior is a uniform drift



Motion of electrons in conductor

N electrons are moving in a material immersed in **E**

- Two components contribute to the momentum:
  - Random collision velocity u_0: $\vec{p}_{Random} = m\vec{u}_0$
  - Impulse due to electric field: $\vec{p}_E = q\vec{E}t$
- The average momentum is:

$$\langle p \rangle = m\langle u \rangle = \frac{1}{N}\sum_{i=1}^{N}(m\vec{u}_i + q\vec{E}t_i) = m\frac{1}{N}\sum_{i=1}^{N}\vec{u}_i + q\vec{E}\frac{1}{N}\sum_{i=1}^{N}t_i$$

- For large N: $\sum_{i=1}^{N}\vec{u}_i \to 0$ → $m\langle u \rangle = \frac{q\vec{E}}{N}\sum_{i=1}^{N}t_i \equiv q\vec{E}\tau$
- Where $\tau \equiv \frac{1}{N}\sum_{i=1}^{N}t_i$ is the average time between 2 collisions
  - Property of the material

Conductivity

- From this derivation we can read off the conductivity

$$\begin{cases} \vec{J} = nq\langle\vec{u}\rangle \\ m\langle\vec{u}\rangle = q\vec{E}\tau \end{cases} \Rightarrow \vec{J} = nq\frac{q\vec{E}\tau}{m} = \sigma\vec{E} \Rightarrow \boxed{\sigma = \frac{nq^2\tau}{m}}$$

- For multiple carriers:

$$\sigma = \sum_{i=1}^{N}\frac{n_k q_k^{\,2}\tau_k}{m_k}$$