

8.022 Electricity and Magnetism  
REVIEW OF CLASSICAL MECHANICS

**(1) Kinematics**

**(a) Position vector, Description of the motion**

We describe the position of a particle by specifying its coordinates with respect to a frame  $S$ .  $S$  is really a point, which serves as the origin, i.e. the point with coordinates  $(0,0,0)$ . Thus, for a Cartesian coordinate system, a particle at a point  $P$  is described by three numbers,  $(x,y,z)$ , the distances of  $P$  from the origin along the three axes. The vector from the origin to  $P$  is called the *position vector*,  $\vec{r}$ , of the particle. The position vector contains all the information regarding this particle. If we know  $\vec{r}$  as a function of time, then we know everything we always wanted to know about this particle's future. Let  $\vec{r} = \vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$ . Then the velocity,  $\vec{v}(t)$ , and the acceleration,  $\vec{a}(t)$ , of the particle at time  $t$  are given by

$$\vec{v}(t) = \frac{d}{dt}\vec{r} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \Rightarrow |\vec{v}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v} = \frac{d^2}{dt^2}\vec{r} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z} \Rightarrow |\vec{a}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

(We use the notation  $\dot{w} \equiv \frac{dw}{dt}$  i.e. the dot indicates the derivative with respect to time)

Inverting the above equations, we can get the position vector if we know the acceleration or the velocity as a function of time. So, assuming we know the velocity,  $\vec{v}(t)$ , as a function of time,

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(t') dt'$$

If we know the acceleration,  $\vec{a}(t)$ , then the velocity is given by

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(t') dt'$$

from which we can also get the position as a function of time:

$$\vec{r}(t) = \vec{r}(0) + \vec{v}(0)t + \int_0^t \int_0^{t'} \vec{a}(t'') dt'' dt'$$

To summarize: we can get the position vector of a particle (if we know its velocity or acceleration as a function of time) by a simple integration, if we know the initial conditions, i.e. the velocity and position vector at some point in time (say  $t=0$ ).

### **(b) Motion in a straight line**

The above equations for the position vector can be applied to the case where the particle is moving in a straight line. There are a few useful special cases:

(a) particle moving with constant velocity in the vacuum (i.e. no friction)

$\vec{r}(t) = \vec{r}_o + \vec{v}t$  where  $\vec{r}_o$  is the initial position of the particle. In one-dimensional motion, say along the x-axis, this equation becomes  $x(t) = x_o + vt$ , i.e. our familiar equation from elementary kinematics.

(b) particle moving with constant acceleration

$\vec{v}(t) = \vec{v}_o + \vec{a}t$  and  $\vec{r}(t) = \vec{r}_o + \vec{v}_o t + \frac{1}{2} \vec{a}t^2$ , where  $\vec{r}_o$  and  $\vec{v}_o$  are the initial position and velocity of the particle, respectively. Again, for motion only along the x-axis, we get:

$$v(t) = v_o + at \text{ and } x(t) = x_o + v_o t + \frac{1}{2} at^2$$

Deceleration is a negative acceleration. Example: a car is moving with constant speed  $u$ . At time  $t=0$  the driver uses the brakes to decelerate uniformly. The car comes to a halt after a distance  $s$ . What was the deceleration of the car?

The velocity of the car, and the distance it has traveled at time  $t$  are given by

$$v = u - at \quad \text{and} \quad x = ut - \frac{1}{2} at^2$$

We are given the total distance it travels until it stops, i.e. until  $v = 0$ . From the velocity equation, this will take  $t=u/a$  sec. Substituting this value of time in the distance equation, we obtain

$$s = u \frac{u}{a} - \frac{1}{2} a \frac{u^2}{a^2} = \frac{u^2}{2a} \Rightarrow a = \frac{u^2}{2s}.$$

Another example: a gun fires a particle at an angle  $\mathbf{q}$  with respect to the horizontal, with initial velocity  $V_o$ . How far from the gun does the bullet land?

Decompose the motion along two independent axes, the horizontal ( $x$ ) and vertical ( $y$ ). We take the origin to be at the gun.

Along the  $x$ -axis: motion with constant velocity  $V_x = V_o \cos \mathbf{q} \Rightarrow x = V_x t$ . Thus, we need to find how long it will take the bullet to strike the ground. This is given by twice the time it takes it to reach its maximum height,  $H$ . Now along the  $y$ -axis, we have a particle that is moving with constant acceleration  $-g$ , and with initial velocity  $V_y = V_o \sin \mathbf{q}$ . We have already computed the time it will take the bullet to reach this maximum height (i.e. the time it will take to stop moving in the vertical direction) in the example above with the decelerating car:  $t = \frac{V_y}{g}$ . Thus, the total travel time is twice that (it takes the

bullet this much longer to come back down to the ground) and thus the horizontal distance from the gun will be

$$x = V_x t = V_x 2 \frac{V_y}{g} = 2 \frac{V_o^2 \sin \mathbf{q} \cos \mathbf{q}}{g} = \frac{V_o^2 \sin 2\mathbf{q}}{g}$$



**(c) Motion in a circle**

Since velocity is a vector, one can have a change in velocity without a change in speed (i.e. the magnitude of the velocity vector). A particle moving with constant speed in a circle of radius  $R$  accelerates continually!

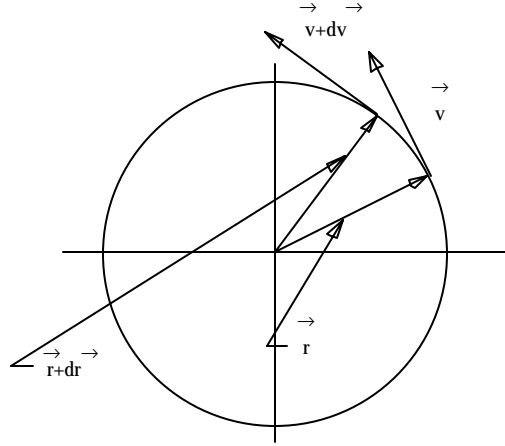
With our origin at the center of the circle, the position vector at time  $t$  and  $t + dt$  is given by  $\vec{r}$  and  $\vec{r} + d\vec{r}$  respectively. The velocity vector is  $\vec{v}$  and  $\vec{v} + d\vec{v}$ . To find the acceleration, we need to compute the vector  $d\vec{v}$ .

From the figure,  $d\vec{v} \approx -v d\hat{f}$ . Therefore,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-v d\hat{f}) = -v \hat{r} \frac{d\hat{f}}{dt} = -v \omega \hat{r}.$$

Finally, since  $v = \omega R$ , we get, for a particle moving in a circle with constant speed  $v$  (or angular velocity  $\omega$ )

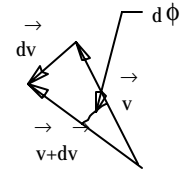
$$\vec{a} = -\frac{v^2}{R} \hat{r} = -\omega^2 R \hat{r}$$



We could have proven this by using straightforward derivatives also:

$$\vec{r}(t) = R(\cos \omega t \hat{x} + \sin \omega t \hat{y}) \Rightarrow \vec{v}(t) = \frac{d\vec{r}}{dt} = R(-\omega \sin \omega t \hat{x} + \omega \cos \omega t \hat{y})$$

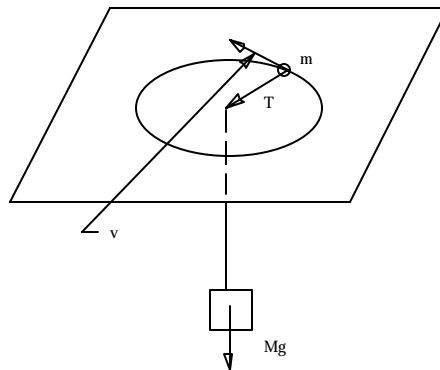
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -R(\omega^2 \cos \omega t \hat{x} + \omega^2 \sin \omega t \hat{y}) \Rightarrow \vec{a}(t) = -\omega^2 \vec{R}$$



Example: a particle of mass  $m$  moving on a frictionless table with constant speed  $v$ . It is connected to a string supporting a mass  $M$ . What is the radius,  $R$ , of the circle?

The particle is moving in a circle, thus it is accelerating, and thus there must be a net force acting on it: the tension from the string,  $T$ .

$$\text{Thus: } T = ma = -m \frac{v^2}{R} = -Mg \Rightarrow R = \frac{mv^2}{Mg}.$$



## **(2) Forces**

### **(a) Newton and Gravity**

According to Newton, force is the "thing" that causes the momentum of an object to change:

$\vec{F} = \frac{d}{dt}(\vec{p}) \Rightarrow \vec{p} = \text{Constant}$  when  $\vec{F} = 0$ . From this, and for the special case of a constant mass, we

get the familiar Force = mass×acceleration formula:

$$\vec{F} = \frac{d}{dt} \vec{p} = \frac{d}{dt} m \vec{v} = m \frac{d}{dt} \vec{v} = m \vec{a}$$

While we are on the subject of Newton, let us remember his third law also, i.e. that Action = Reaction:  $\vec{F}_{21} = -\vec{F}_{12}$ , i.e. the force on particle 2 from particle 1,  $\vec{F}_{21}$ , is equal and opposite to the force on particle 1 from particle 2,  $\vec{F}_{12}$ .

A most familiar example of a force is gravity. The gravitational force between two bodies with masses  $m_1$  and  $m_2$  is  $F = -G \frac{m_1 m_2}{r^2}$  where the  $-$  sign indicates that the force is attractive. In the case of the earth and an apple, which is at a height  $h$  above the earth's surface, the magnitude of the force is given by

$$F = G \frac{M m_a}{(R + h)^2} = \frac{G M m_a}{R^2} \left(1 + \frac{h}{R}\right)^{-2} \cong \frac{G M m_a}{R^2} \left(1 - \frac{2h}{R}\right) \text{ where } R \text{ is the Earth's radius.}$$

A typical apple tree is about a few meters tall, and the radius of the earth is approximately 6,400 km. Thus, the second term in the parenthesis is of the order one millionth ( $10^{-6}$ ), and is thus negligible compared to the first term. Thus,  $F = \frac{G m_e}{R^2} m_a = m_a g$  where  $g$  is the familiar (constant) acceleration due to gravity.

Equipped with the kinematics from section 1, we can now solve for the motion of any particle in the gravitational field: it corresponds to motion with constant acceleration  $-g$  along the  $y$ -axis. We have already used this in the section on kinematics in the ballistics problem.

### **(b) Conservation of momentum and Collisions.**

If the total external force on a system of particles is zero, then the total momentum of the system is constant. Applications: two colliding masses  $m_a$  and  $m_b$  with initial velocities  $v_a$  and  $v_b$  respectively, collide head-on, and stick to each other. What is the velocity of the two masses after the collision?

$m_a v_a + m_b v_b = (m_a + m_b) V$  where  $V$  is the wanted final velocity.

### (3) Work and Energy

#### (a) Introduction

Suppose we know the total force,  $\vec{F}(\vec{r}, t)$ , acting on a particle of mass  $m$ , as a function of the particle's position and time. Classical mechanics addresses the problem of predicting the motion of this particle: given this force and some initial conditions (e.g. the position and velocity of the particle at some time previous time) what is the velocity and position of this particle?

Let's look at this problem in the one-dimensional case first, and let's assume that the force does not vary with time, i.e.  $F = F(x)$ . To solve for the motion, we integrate with respect to  $x$  :

$$F(x) = m \frac{dv}{dt} \Rightarrow \int_{x_a}^{x_b} F(x) dx = m \int_{x_a}^{x_b} \frac{dv}{dt} dx = m \int_{t_a}^{t_b} \frac{dv}{dt} v dt = m \int_{t_a}^{t_b} \frac{d}{dt} \left( \frac{1}{2} v^2 \right) dt = \frac{1}{2} m (v_b^2 - v_a^2)$$

Thus, if we know the velocity,  $v_o$ , at some position  $x_o$ , the velocity,  $v$ , at position  $x$  is given by  $\frac{1}{2} mv^2 = \frac{1}{2} mv_o^2 + \int_{x_o}^x F(x) dx$ . Since we have the velocity, we can now find the position as a function of time by integrating the velocity with respect to time.

Nomenclature:

$\frac{1}{2} mv^2 =$  Kinetic Energy,  $K$ , of the particle;

$\int_{x_a}^{x_b} F(x) dx =$  work  $W_{ba}$  done by the force  $F$  on the particle as it moves from  $a$  to  $b$ .

Work-Energy Theorem:

$$K_b - K_a = W_{ba}$$

Generalize this to three-dimensional motion, where the particle displacement in an infinitesimal time  $dt$  is not just  $dx$  but  $d\vec{r}$ . Forming the dot (also called "scalar") product between the vectors  $\vec{F}$  and  $d\vec{r}$  and integrating, we get

$$\vec{F}(\vec{r}) = m \frac{d\vec{v}}{dt} \Rightarrow \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r}) \cdot d\vec{r} = m \int_{\vec{r}_a}^{\vec{r}_b} \frac{d\vec{v}}{dt} \cdot d\vec{r} = m \int_{t_a}^{t_b} \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_{t_a}^{t_b} \frac{d}{dt} \left( \frac{1}{2} v^2 \right) dt = \frac{1}{2} m (v_b^2 - v_a^2)$$
 i.e. the

same result like before, that the work done by the force is equal to the change in kinetic energy of the object.

Note how the work is now a funny integral, namely it is not a usual integral like  $\int_a^b f(x) dx$  but rather, the integral of a vector function dotted with the infinitesimal displacement, over the path followed by the particle, i.e.  $\int_a^{b \rightarrow} \vec{f}(\vec{r}) \cdot d\vec{r}$ . This is our first example of a *line integral*. It is different from a normal integral in that here we have to evaluate the integrand along a particular path that joins points  $a$  and  $b$ .

Example: energy of a particle of mass  $m$  in a gravitational field. The field is generated by an object of mass  $M$ . The work that we do in transporting the particle from infinity (where it is free of all forces) to a

distance  $r$  from  $M$  is  $W = -\int_{\infty}^r \vec{F}_G(\vec{r}) \cdot d\vec{r}$  where the minus sign accounts for the fact that the force we exert is opposite to the gravitational force between the two masses. Then,  $W = -\int_{\infty}^r -\frac{GMm}{r^2} dr = -\frac{GMm}{r}$ . This work appears as the *Potential Energy* of the particle  $m$ .  $W$  is independent of the path we use in transporting  $m$  from infinity to the point  $P$ . For the apple at a height  $h$  above the earth's surface,

$$W = -\frac{GMm}{R+h} = -\frac{GMm}{R} \left(1 + \frac{h}{R}\right)^{-1} \approx -\frac{GMm}{R} \left(1 - \frac{h}{R}\right) = -\frac{GMm}{R} + \frac{GMm}{R^2} h = \text{Constant} + mgh$$

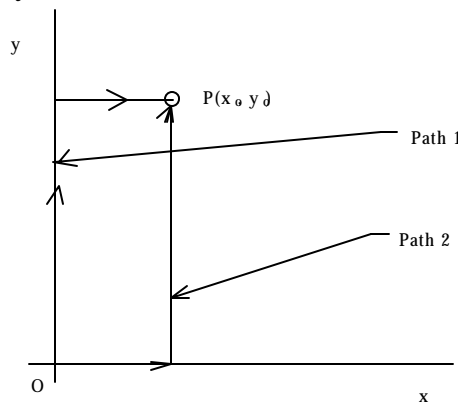
In other words, apart from a constant factor, Potential Energy =  $mgh$

### **(b) Line Integrals, Work–Energy theorem**

In general, the line integral of a vector function  $\vec{A}$  along a path of integration,  $C$  is written as  $\int_C \vec{A} \cdot d\vec{l}$ .

Here  $d\vec{l}$  denotes the element of path along  $C$ . Remarkable property of gravitational field:  $\int_{C_1} \vec{F}_G \cdot d\vec{l} = \int_{C_2} \vec{F}_G \cdot d\vec{l}$ , i.e. the work done in transporting a particle of any mass from point  $A$  to point  $B$  is independent of the path used. The gravitational field is an example of a *Conservative Field*.

This independence of the line integral on the path  $C$  is not a property of all vector fields. As an example, suppose a particle of mass  $m$  is acted upon by a force which is a function of position on the  $x$ - $y$  plane:  $\vec{F}(x, y) = x^2 y \hat{x} + xy^2 \hat{y}$ .



If the particle moves from the origin to point  $P = (x_0, y_0)$  via path 1 as shown in the figure, then the work done along this path,  $W_1$ , is  $W_1 = \int_0^{x_0} x^2 y_0 dx = \frac{1}{3} x_0^3 y_0$ . However, if we compute the work done by the same force,  $W_2$ , along path 2, then  $W_2 = \int_0^{y_0} x_0 y^2 dy = \frac{1}{3} x_0 y_0^3$ . i.e.  $W_2 \neq W_1$ . The field  $\vec{F}$  in this case is not a conservative field.



When the field is conservative, the line integral from point A to point B is dependent only on the position of these two points, i.e.  $-\int_C \vec{F} \cdot d\vec{r} = U(r_B) - U(r_A)$ . In other words, there exists a scalar function  $U(r)$  which is given by  $U(r) = U_o - \int_C \vec{F} \cdot d\vec{r}$  where C is any path that joins the reference point O and the point P at radial position r. Using the work–energy theorem,

$$\boxed{K_B - K_A = W_{BA} = -(U_B - U_A) \Rightarrow K_B + U_B = K_A + U_A = \text{Constant}}$$

This constant we call the total mechanical energy of the particle,  $E$ . It is a constant of the motion:

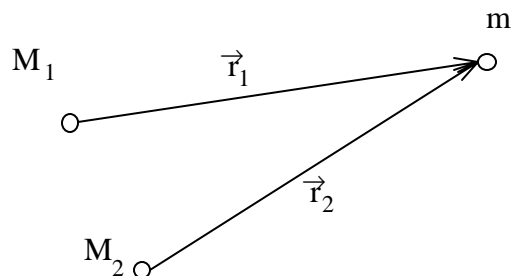
$$\boxed{E = K + U = \text{Constant}}$$

### (c) The idea of a field; Potential

The gravitational field due to a sphere of mass  $M$  and radius  $R$ , for distances  $r > R$ , is given by

$$\vec{E}_G = -\frac{GM}{r^2} \hat{r} \quad \text{and the force on a mass } m \text{ is } \vec{F} = m\vec{E}_G$$

Note that in the above equation the field due to the mass  $M$  is independent of any other field sources! Thus, for any collection of point masses, we can claim that we know the total field due to gravity — by simply applying the superposition principle. As an example, given two masses a and b, what is the field at point  $m$ ?



The field at the point  $m$  is the sum of the fields due to each mass separately:

$$\begin{aligned} \vec{E}_G(\text{Total}) &= \vec{E}_G(M_1) + \vec{E}_G(M_2) \\ &= -\frac{GM_1}{r_1^3} \vec{r}_1 - \frac{GM_2}{r_2^3} \vec{r}_2 \end{aligned}$$

The force on  $m$  is thus  $\vec{F} = m\vec{E}_G(\text{Total})$ .

The moral of the story is: given the field at any point, we can compute the force on any small "charge" of mass  $m$  at that point. Given that we know how to get from the force to the velocity of the test charge (i.e. through the work–energy theorem) and then from the velocity to the position of the test charge as a function of time, we conclude that all we need to do to solve for the motion of a small test charge  $m$  is to find the gravitational field at all points in space. (Note: this is true only for the case where the mass  $m$  is itself very small or the field sources are fixed magically).

Having defined the field in this way, we can now define the *Potential* of the field,  $\mathbf{f}$ , so that the potential energy,  $U$ , of a particle of mass  $m$  in the field is given by  $U = m\mathbf{f}$ . Just like the field is the force per unit charge, the potential is the potential energy per unit charge. (Charge in this case refers to mass, i.e. the "charge" of the gravitational field). So what is left now? To establish a relationship for calculating the field (and thus the force also) given the potential of a field. Since

$\mathbf{f} = \frac{U}{m} = -\int_C (\vec{F} / m) \cdot d\vec{r} = -\int_C \vec{E}_G \cdot d\vec{r}$ , we conclude that the potential of the field is the line integral of

the field (let us not bother for the moment with the reference point, i.e. the point for which we know the value of the potential). Then the question is, how do we get the field if we know the potential?

For an infinitesimal path, the line integral is equal to  $-\vec{E}_G \cdot d\vec{r} = -E_{G,x}dx - E_{G,y}dy - E_{G,z}dz = d\mathcal{F}$ . But

we know that  $\mathcal{F} = \mathcal{F}(x, y, z) \Rightarrow d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial x} dx + \frac{\partial \mathcal{F}}{\partial y} dy + \frac{\partial \mathcal{F}}{\partial z} dz$ . Therefore,

$$\boxed{E_{G,x} = -\frac{\partial \mathcal{F}}{\partial x}; E_{G,y} = -\frac{\partial \mathcal{F}}{\partial y}; E_{G,z} = -\frac{\partial \mathcal{F}}{\partial z}}$$

And since  $F_x = mE_x$ ;  $F_y = mE_y$ ;  $F_z = mE_z$ , we get

$$\boxed{\vec{F} = -\frac{\partial U}{\partial x} \hat{x} - \frac{\partial U}{\partial y} \hat{y} - \frac{\partial U}{\partial z} \hat{z}}$$

In other words, given the potential energy of a particle due to an external field, we can compute the force on this particle. Example: for the gravitational field,  $U = -\frac{GMm}{r}$ . Then,

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x} \left( -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{GMmx}{(x^2 + y^2 + z^2)^{3/2}} = \frac{GMm}{r^3} x.$$

From a similar calculation for the y and z components, we finally get

$$\vec{F} = -\frac{GMm}{r^3} (x\hat{x} + y\hat{y} + z\hat{z}) = -\frac{GMm}{r^3} \vec{r} = -\frac{GMm}{r^2} \hat{r},$$

i.e. we recover our familiar gravitational force.

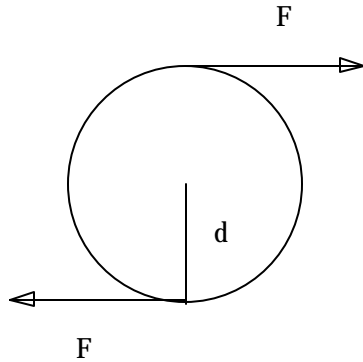
The final step is to recognize that using the gradient operator, we write all of the above in a nice shorthand notation, namely,

$$\boxed{\vec{F} = -\vec{\nabla} U \quad \text{and} \quad \vec{E} = -\vec{\nabla} \mathcal{F}}$$

Details on the gradient can be found in the appendix.

#### (4) Angular momentum, Torque

The idea is simple: there are cases in which the total force on an object is zero, yet the object is accelerating.



An example is given in the figure, where two forces are applied to a cylinder, attached to a frictionless axis through its center. It is clear that the cylinder will rotate with respect to its axis.

We define the angular momentum of a particle with respect to an axis to be  $\vec{L} = \vec{r} \times \vec{p}$ , where  $\vec{r}$  is the vector from the axis to the particle, and  $\vec{p}$  is the momentum of the particle.

The magnitude of the angular momentum of a particle is thus

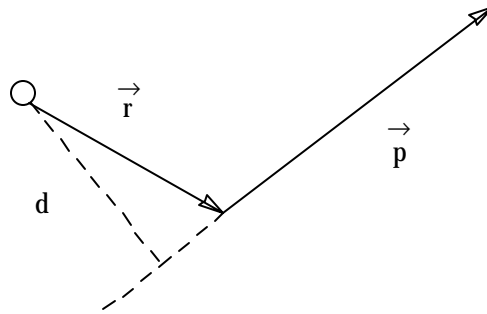
$$|\vec{L}| = rp \sin \theta = pd$$

where  $d$  is the perpendicular distance from the axis to the momentum of the particle.

The torque with respect to an axis is

defined in a similar way:  $\vec{\tau} = \vec{r} \times \vec{F}$ . But then:

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}.$$



There is one more quantity that we need to complete the discussion of rotations of rigid bodies, that of the moment of inertia. Suppose a body is rotating with angular velocity  $\omega$  with respect to some fixed axis O. Then, the total kinetic energy of the body is given by the sum of the kinetic energies of the infinitesimal particles into which we can divide it:

$$E_k = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{2} m_i v_i^2 = \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{2} m_i r_i^2 \omega^2 = \left( \lim_{N \rightarrow \infty} \sum_{i=1}^N \frac{1}{2} m_i r_i^2 \right) \omega^2$$

The quantity in parentheses clearly depends only on the geometrical distribution of the mass on the body. The sum can be turned into an integral:

$$I = \int r^2 dm = \int r^2 \rho dV, \text{ where } \rho \text{ is the density of the body and } dV \text{ the element of volume.}$$

But then, the kinetic energy of the rigid body is given by  $E_k = \frac{1}{2} I \omega^2$ , whereas the angular momentum is

$$\text{given by } \vec{L} = \lim_{N \rightarrow \infty} \sum_{i=1}^N \vec{r}_i \times \vec{p}_i = \lim_{N \rightarrow \infty} \sum_{i=1}^N \vec{r}_i \times (m_i \vec{\omega} \times \vec{r}_i)$$

Utilizing  $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$ , we finally get:

$$\vec{L} = \lim_{N \rightarrow \infty} \sum_{i=1}^N m_i \{ (\vec{r}_i \cdot \vec{r}_i) \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \vec{r}_i \} = \left( \lim_{N \rightarrow \infty} \sum_{i=1}^N m_i r_i^2 \right) \vec{\omega} \Rightarrow \vec{L} = I \vec{\omega}$$

In other words, there is a one-to-one correspondence between translations and rotations: mass  $\rightarrow$  moment of inertia, velocity  $\rightarrow$  angular velocity, and force  $\rightarrow$  torque. Summarizing:

	Translation		Rotation	
position	$x$	$\rightarrow$	angle	$\mathbf{q}$
velocity	$v$	$\rightarrow$	angular velocity	$\mathbf{w}$
acceleration	$a$	$\rightarrow$	angular acceleration	$\mathbf{a}$
force	$F$	$\rightarrow$	torque	$\mathbf{t}$
momentum	$p$	$\rightarrow$	angular momentum	$L$
mass	$m$	$\rightarrow$	moment of inertia	$I$

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$$v = \frac{dx}{dt} \rightarrow \mathbf{w} = \frac{d\mathbf{q}}{dt}$$

$$a = \frac{dv}{dt} \rightarrow \mathbf{a} = \frac{d\mathbf{w}}{dt}$$

$$\vec{p} = m\vec{v} \rightarrow \vec{L} = I\vec{w}$$

$$\vec{F} = m\vec{a} \rightarrow \vec{t} = I\vec{a}$$

$$\vec{F} = \frac{d\vec{p}}{dt} \rightarrow \vec{t} = \frac{d\vec{L}}{dt}$$

$$E_t^k = \frac{1}{2}mv^2 \rightarrow E_r^k = \frac{1}{2}I\mathbf{w}^2$$

## (A) Multivariable Calculus

### (1) Scalar Functions

For a function  $f(x)$  of one variable  $x$ , the derivative is defined as

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

we know that the derivative at a point is the slope, i.e. the rate of change of the function with respect to  $x$ . In other words, we can predict the change in the value of the function  $f(x)$  in the immediate vicinity of a point  $x_0$ , provided we have the derivative of  $f$  at that point:

$$df = \left( \frac{df}{dx} \right)_{x_0} dx = \left[ \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right] dx$$

What if the function  $f$  is a function of two variables,  $x$  and  $y$ , i.e.  $f=f(x,y)$ ? Given a point  $(x_0, y_0)$ , we would like to know the change in the function when we change simultaneously  $x$  by  $dx$  and  $y$  by  $dy$ .

To compute this change, we first keep the variable  $y$  fixed at the point  $y_0$  and let the variable  $x$  change by  $dx$ ; the change in  $f$ ,  $df$ , is given by

$$df = \left[ \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \right] dx$$

Similarly, if we keep  $x$  fixed at  $x_0$  and we change  $y$  by  $dy$ , the change in  $f$  is given by

$$df = \left[ \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \right] dy$$

we thus define a partial derivative for the function  $f(x,y)$  with respect to one of the variables  $x$  or  $y$  as the derivative of  $f$  with respect to  $x$  or  $y$ , keeping the other variable ( $y$  or  $x$ ) fixed. We denote this partial derivative by  $\frac{\partial f}{\partial x}$  or  $\frac{\partial f}{\partial y}$ .

Example 1: suppose that  $f(x,y) = xy$ . Then,  $\frac{\partial f}{\partial x} = y$  and  $\frac{\partial f}{\partial y} = x$

Example 2:  $f(x,y) = 3x^2y + 2y^3 \Rightarrow \frac{\partial f}{\partial x} = 6xy$  and  $\frac{\partial f}{\partial y} = 3x^2 + 6y^2$

Equipped with all this, we can now ask what happens when both  $x$  and  $y$  change by  $dx$  and  $dy$  respectively:

$$\begin{aligned} \Delta f &= f(x + \Delta x, y + \Delta y) - f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y) \\ &= \Delta f(y \text{ constant at } y + \Delta y) + \Delta f(x \text{ constant at } x) \\ &= \frac{\partial f(x, y + \Delta y)}{\partial x} \Delta x + \frac{\partial f(x, y)}{\partial y} \Delta y \end{aligned}$$

taking the limit of infinitesimal  $\mathbf{Dx}$  and  $\mathbf{Dy}$ ,

$$df = \lim_{\Delta x \rightarrow 0, \Delta y \rightarrow 0} \Delta f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

The generalization to a function of three variables,  $f(x, y, z)$ , is obvious:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

## (2) Vector Functions

Let's look at a vector function,  $\vec{A}(p)$ , which depends on only one variable,  $p$ . What is the change in this function as  $p$  changes by  $dp$ ? We realize that this vector function is nothing more than three independent functions  $A_x(p), A_y(p), A_z(p)$ . Then,

$$\begin{aligned} d\vec{A} &= \vec{A}(p+dp) - \vec{A}(p) \\ &= (A_x(p+dp) - A_x(p))\hat{x} + (A_y(p+dp) - A_y(p))\hat{y} + (A_z(p+dp) - A_z(p))\hat{z} \\ &= dA_x\hat{x} + dA_y\hat{y} + dA_z\hat{z} \end{aligned}$$

Now since  $A_x(p), A_y(p), A_z(p)$  are scalar functions of only one variable, we already know how to evaluate their differentials, namely  $dA_i = \left( \frac{dA_i}{dp} \right)_p dp$ ,  $i \in \{x, y, z\}$

The final answer is thus

$$d\vec{A} = \left( \frac{dA_x}{dp} \hat{x} + \frac{dA_y}{dp} \hat{y} + \frac{dA_z}{dp} \hat{z} \right) dp$$

Example: the position vector of a particle is given by  $\vec{r} = 3t^2\hat{x} + 6t\hat{y} + (3-4t)\hat{z}$ . We have been through this before: the change in the position vector due to an infinitesimal change in time is simply  $d\vec{r} = (6t\hat{x} + 6\hat{y} - 4\hat{z})dt$ .

Finally, we investigate vector functions of several variables. For illustration purposes, we consider a three-dimensional vector function of two variables, say  $p$  and  $q$ :

$$\vec{A}(p, q) = A_x(p, q)\hat{x} + A_y(p, q)\hat{y} + A_z(p, q)\hat{z} \Rightarrow d\vec{A} = dA_x(p, q)\hat{x} + dA_y(p, q)\hat{y} + dA_z(p, q)\hat{z}$$

and the differential of the  $i$ th component ( $i = x, y, z$ ) is given by

$$dA_i(p, q) = \frac{\partial A_i}{\partial p} dp + \frac{\partial A_i}{\partial q} dq$$

We now know how to compute the variation of any function, scalar or vector, i.e. 1-D or N-D (where 2-D for example, means 2-dimensional), with respect to variations in the parameters the function depends on.

A very interesting — and physically meaningful — case is when one has a vector function that depends on position. An example is the gravitational field. A mass  $M$  creates a field around it. Any mass  $m$  brought close to  $M$  will thus experience a force. We define the field so that it is dependent only on parameters associated with the source. In other words, we define  $\vec{E}_G = -\frac{GM}{r^2}\hat{r}$ . Given the field, the

force on any mass  $m$  is given by  $\vec{F}_G = m\vec{E}_G = -\frac{GMm}{r^2}\hat{r}$ . In Cartesian coordinates, the field is written as

$$\vec{E}_G = -\frac{GM}{(x^2 + y^2 + z^2)^{3/2}}(x\hat{x} + y\hat{y} + z\hat{z})$$

so this is a vector field which depends on three variables,  $x$ ,  $y$  and  $z$ . The change in this field in the neighborhood of  $(x, y, z)$  i.e. when  $x \rightarrow x + dx$ ,  $y \rightarrow y + dy$ ,  $z \rightarrow z + dz$  is then

$$d\vec{E}_G = dE_x\hat{x} + dE_y\hat{y} + dE_z\hat{z} \text{ with } dE_i = \frac{\partial E_i}{\partial x} dx + \frac{\partial E_i}{\partial y} dy + \frac{\partial E_i}{\partial z} dz$$

$$dE_x = -GM \frac{(y^2 + z^2 - 2x^2)dx - 3xydy - 3xzdz}{(x^2 + y^2 + z^2)^{5/2}}$$

$$dE_y = -GM \frac{-3xydx + (z^2 + x^2 - 2y^2)dy - 3yzdz}{(x^2 + y^2 + z^2)^{5/2}}$$

$$dE_z = -GM \frac{-3xzdx - 3yzdy + (x^2 + y^2 - 2z^2)dz}{(x^2 + y^2 + z^2)^{5/2}}$$

### (3) The gradient operator

An operator is a "thing" that acts on an object. In this sense, the derivative operator is equal to  $\frac{d}{dx}$ .

Given any function  $f(x)$ , this operator acts on the function to deliver a new function, the derivative of  $f(x)$ . Thus,  $\frac{d}{dx} f(x) = \frac{df}{dx}$ .

By the same token, we can think of the differential of a function of three variables as the dot product of two vectors: the vector  $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$  and the vector  $(dx, dy, dz)$ . Let us call the first vector  $\vec{C}$  — the

second vector being equal to  $d\vec{r}$ . Then,  $df = \vec{C} \cdot d\vec{r}$ . In other words, for any function  $f(x, y, z)$  we can define the vector  $\vec{C} \equiv \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)f$

We define an operator to be equal to the parenthesis that acts on  $f$ :

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

we call this operator the gradient operator. When it acts on a function, it gives as a vector whose direction is the direction of maximum change of the function. Its magnitude is equal to the rate of change of the function, in this direction of maximum change, with respect to changes in  $x, y$  and  $z$ .

This is but a generalization of everything we have done so far: The change in a function of one variable,  $f(x)$ , due to change in  $x \rightarrow x + dx$  is  $df = \frac{df}{dx} dx = \frac{df}{dx} \hat{x} \cdot dx \hat{x}$ . For a function of three variables,  $f(x, y, z)$ , the equivalent change is  $df = \vec{C} \cdot d\vec{r}$ , where  $\vec{C} \equiv \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}$

What is the meaning of the gradient of a function? Remember, the gradient is a vector. The direction of the vector is the direction in which the change in the function is maximum. And the magnitude of the vector is equal to the rate of change of the function in this maximal direction.

Example: a fly is in a room whose temperature is given by  $T(x, y, z) = \frac{z}{x^2 + y^2}$  (i.e. the air is warmer as we go up towards the ceiling, and warmer as we approach, say, the point (0,0) which is taken to be the middle of the room. In which direction should the fly move in order to maximize the change in temperature it will experience? This direction is given by the gradient of the temperature:  $\vec{\nabla} T = -\frac{2xz}{(x^2 + y^2)^2} \hat{x} - \frac{2yz}{(x^2 + y^2)^2} \hat{y} + \frac{z}{(x^2 + y^2)} \hat{z}$ . As expected, this direction is towards the center of the room in the  $x$ - $y$  plane, and upwards in the  $z$  direction.

Finally, in cylindrical coordinates,  $(r, \phi, z)$ , the gradient is given by  $\vec{\nabla} f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$ .