Welcome back to 8.033!

Leonhard Euler, Swiss, 1707-1783

Image Courtesy of Wikipedia

Summary of course so far: See study guide

Main focus: be able to solve problems that involve converting between different inertial frames

MIT Course 8.033, Fall 2006, Lecture 8

Max Tegmark

Today: Geodesics, calculus of variations

- The Euler-Lagrange equation
- Deriving it
- Using it:
  - metrics, Euclidean space geodesics
  - Minkowski space geodesics
  - gravitational redshift
  - brachistochrone problem
  - caternary

Photograph of a brachistochrone experiment. Image removed due to copyright restrictions.

Brachistochrone flicks

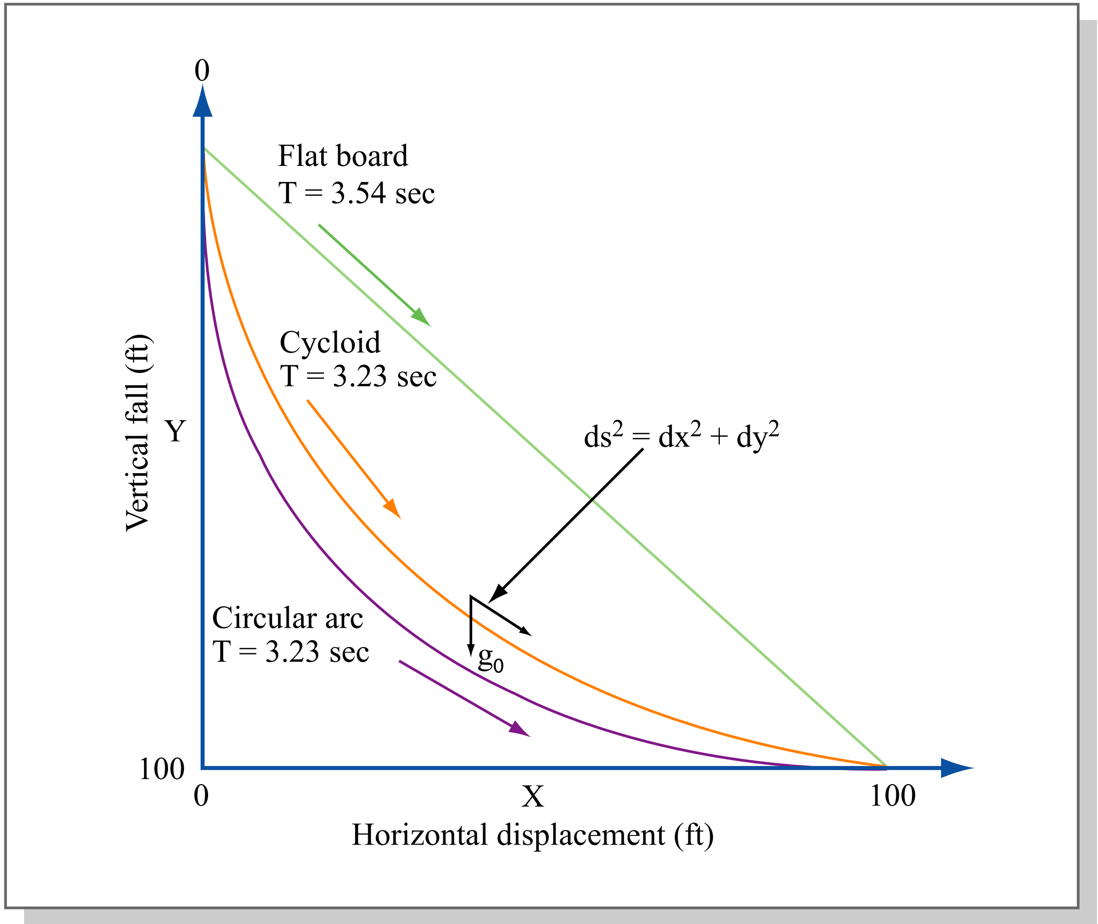

Figure by MIT OCW.

Metrics and geodesics

- In an n-dimensional space, the *metric* is a (usually position-dependent) $n \times n$ symmetric matrix $\mathbf{g}$ that defines the way distances are measured. The length of a curve is $\int d\sigma$, where

$$d\sigma^2 = d\mathbf{r}^t\, \mathbf{g}\, d\mathbf{r},$$

and $\mathbf{r}$ are whatever coordinates you're using in the space. If you change coordinates, the metric is transformed so that $d\sigma$ stays the same ($d\sigma$ is invariant under all coordinate transformations).

- **Example:** 2D Euclidean space in Cartesian coordinates.

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$d\sigma^2 = d\mathbf{r}^t\, \mathbf{g}\, d\mathbf{r} = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2,$$

$$\int d\sigma = \int \sqrt{d\mathbf{r}^t\, \mathbf{g}\, d\mathbf{r}} = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}\, dx.$$

Applying the Euler-Lagrange equation to this shows that the shortest path between any two points is a straight line.

- **Example:** 4D Minkowski space in Cartesian coordinates ($c = 1$ for simplicity)

$$\mathbf{g} = \boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

c = 1

$$\begin{aligned} d\tau^2 &= -d\sigma^2 = d\mathbf{x}^t \, \mathbf{g} \, d\mathbf{x} = \\ &= \begin{pmatrix} dx & dy & dz & dt \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} \\ &= dt^2 - dx^2 - xy^2 - dz^2, \end{aligned}$$

$$\begin{aligned} \Delta\tau &= \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} dt \\ &= \int \sqrt{1 - u^2} dt = \int \frac{dt}{\gamma}. \end{aligned}$$

Applying the Euler-Lagrange equation to this shows that the extremal interval between any two events is a straight line though spacetime.