8.03 Lecture 10

Last time we discussed the wave equation:

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

Normal modes: standing waves!

(1)

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m)$$

(2) There is a special kind of solution:

$$\psi(x,t) = f(x - v_p t)$$

for any functional form of f. Let $\tau \equiv x - v_p t$

$$(i) \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{\partial f}{\partial \tau} \cdot 1 = f'(\tau)$$

$$\frac{\partial^2 f}{\partial x^2} = f''(\tau)$$

$$(ii) \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial t} = -v_p \frac{\partial f}{\partial t} = -v_p f'(\tau)$$

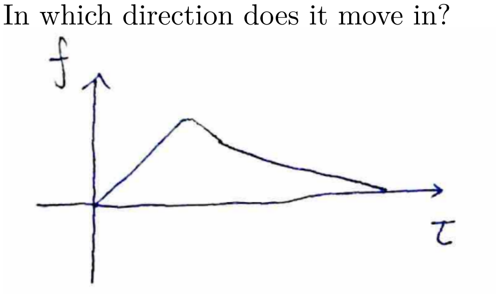
$$\frac{\partial^2 f}{\partial t^2} = v_p^2 f''(\tau)$$

From (i) and (ii):

$$\frac{\partial^2 f}{\partial t^2} = v_p^2 \frac{\partial^2 f}{\partial x^2}$$

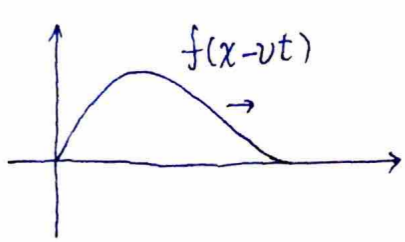
We've learned that $f(\tau)$ satisfies the wave equation! You can also show that any function $f(kx \pm \omega t)$ gives the same result, given $\omega = v_p k$

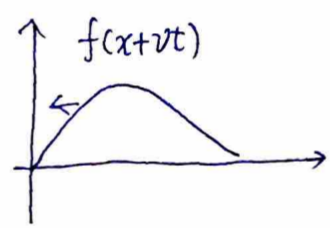
In which direction does it move in?



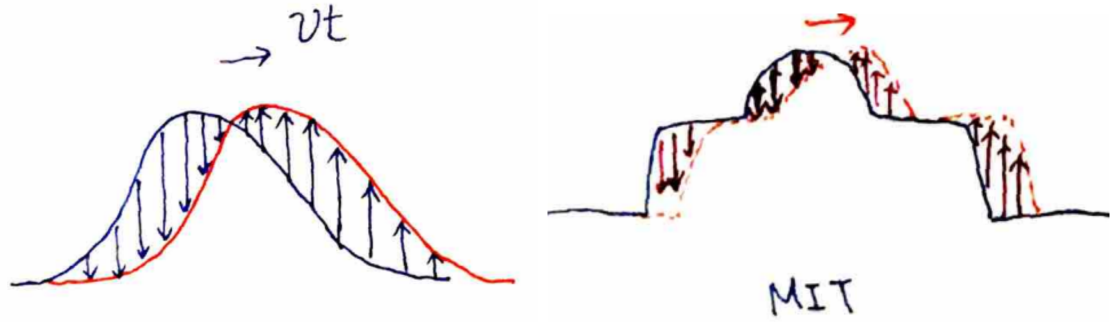
$f(\tau)$ is the shape of the progressing wave

$f(x - vt)$: moving to the right!
$f(x + vt)$: moving to the left!

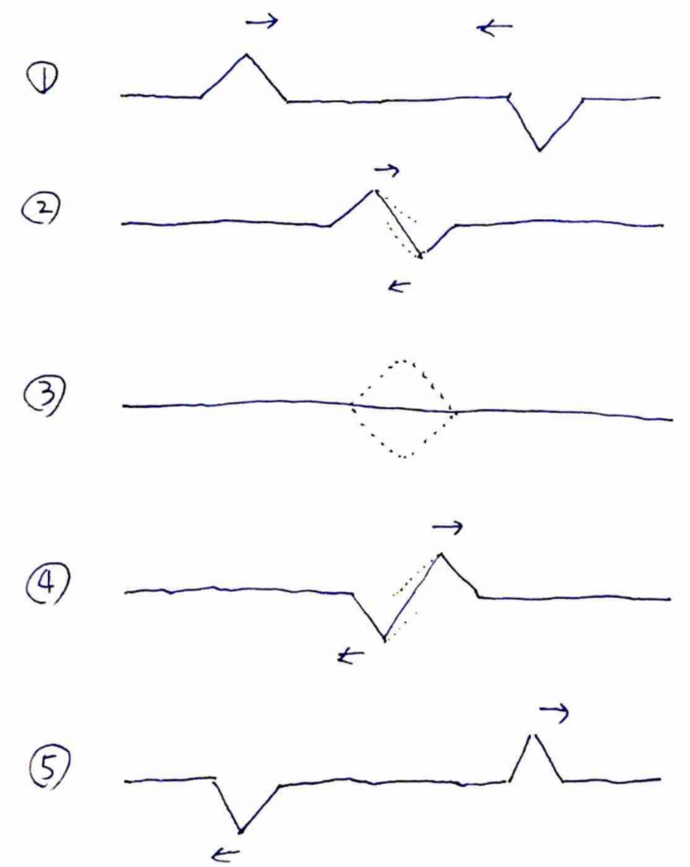




What is moving? The particles in the spring only <u>move up and down!</u>



This works for any shape! This is not obvious, but it is is what is happening.
Wave equations are linear: this means that a linear combination of solutions is a solution.



What will happen? will they cancel? pass through each other or change shape?

Why do they do this? How does the string "remember" what happened?

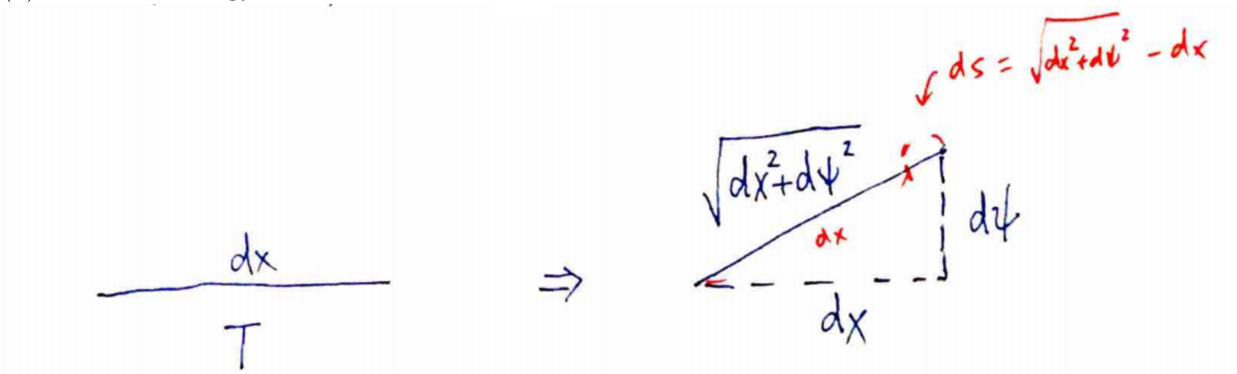
(3) is different from a stationary string: instantaneous velocity in the region where cancellation happens is not zero! I.e. the string is ready to produce the two outgoing progressing waves!
Energy stored in the string:

(1) Kinetic energy $\frac{1}{2}mv^2$

$$\Rightarrow \quad \int \frac{\rho_L}{2} dx \left(\frac{\partial \psi}{\partial t}\right)^2$$

Because $dm = \rho_L dx$ (the differential mass is related to a infinitesimal length element by the density)

(2) Potential energy: $dW = F \cdot ds$



$$\begin{aligned} F \cdot ds \Rightarrow & T\left(\sqrt{dx^2 + d\psi^2} - dx\right) \\ & = T\left(dx\sqrt{1 + \left(\frac{\partial \psi}{\partial x}\right)^2} - dx\right) \end{aligned}$$

We have a small vibration so $\left(\frac{\partial \psi}{\partial x}\right)$ is small

$$= T\left(dx + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 dx - dx\right)$$

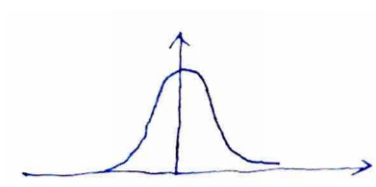
$$\int \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 dx$$

<u>Summary:</u>

$$\begin{cases} \text{Potential energy:} & \int \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 dx \\ \text{Kinetic energy:} & \int \frac{\rho_L}{2} dx \left(\frac{\partial \psi}{\partial t}\right)^2 \end{cases}$$

Example:

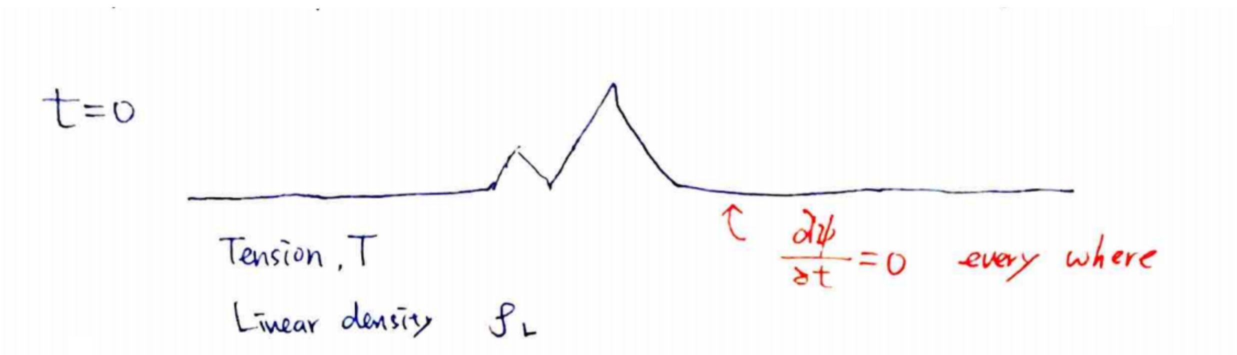
$$\psi(x,t) = \frac{1}{1 + (x - 3t)^4}$$



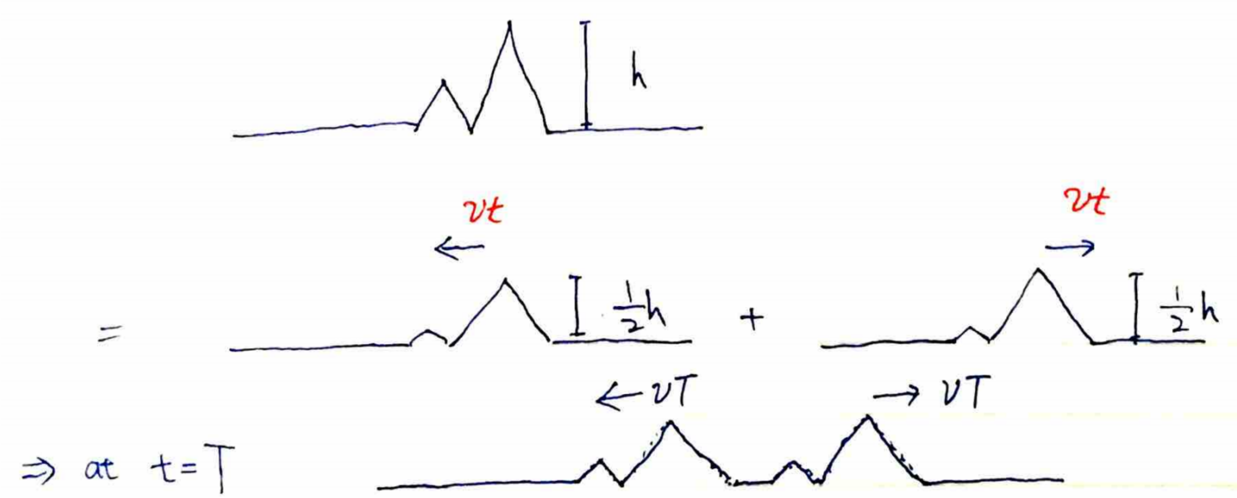
We can write ψ as $f(x-3t)$. The velocity is 3 and it is traveling to the right. Another way to find the velocity is by using the wave equation:

$$v = \sqrt{\frac{\partial^2\psi}{\partial t^2} / \frac{\partial^2\psi}{\partial x^2}}$$

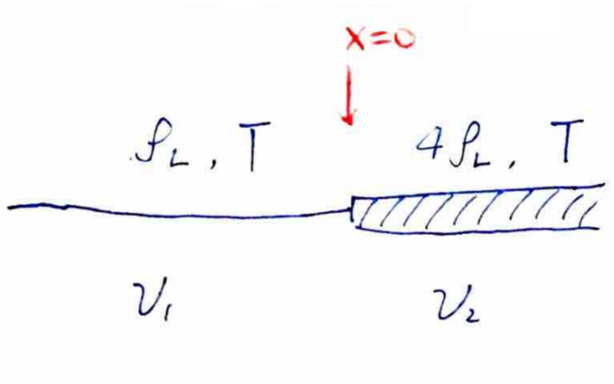
Finally, if you start with a stationary shape:



What will happen at $t=T$? Can we predict? (Define $v \equiv \sqrt{T/\rho_L}$)
(1) Brute force:
Decompose it into ∞ number of normal mode standing waves. Evolve ∞ of those waves
(2) Use $g = f(x+vt) + f(x-vt)$. Velocity : $\frac{\partial g}{\partial t} = vf' - vf'$
*Any stationary shape can be decomposed into two progressing waves!!



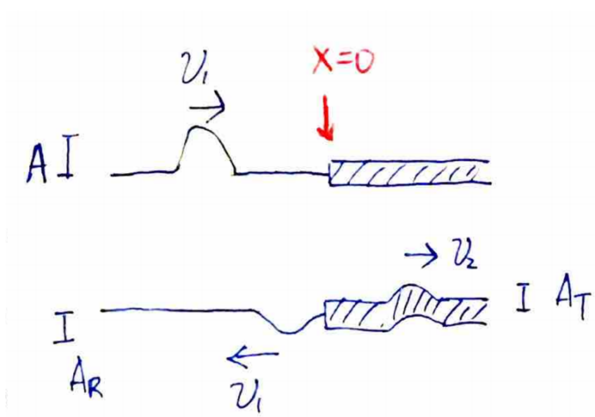
Similarly: normal modes can be decomposed into two traveling sine waves.
A few examples with string: Now we are connecting two systems with different densities, ρ_L and $4\rho_L$



Assuming that the tension, T, is uniform.

$$v_1 = \sqrt{\frac{T}{\rho_L}} \qquad v_2 = \sqrt{\frac{T}{4\rho_L}} = \frac{1}{2}v_1$$

Suppose we have an incident wave with amplitude A



There will be a reflected wave and transmitted wave.
Boundary conditions:

1. The string is continuous: $y_L(0^-) = y_R(0^+)$ ($\Rightarrow \omega$ has to be the same).
2. The slope is continuous:
$$\left.\frac{\partial y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial y_R}{\partial x}\right|_{x=0}$$

If the slope were not continuous there would be a huge acceleration at the junction!

$$\begin{aligned} y_L(x,t) &= f_i(-k_1x+\omega t) + f_r(k_1x+\omega t) \\ y_R(x,t) &= f_t(-k_2x+\omega t) \end{aligned}$$

Where we have the incident, reflected, and transmitted waves.

$$k_1 = \frac{\omega}{v_1} \qquad k_2 = \frac{\omega}{v_2}$$

1. $f_i(\omega t) + f_r(\omega t) = f_t(\omega t)$
2. $-k_1 f_i'(\omega t) + k_1 f_r'(\omega t) = -k_2 f_t'(\omega t)$
   integration on both sides, replace k by v
   $\Rightarrow \quad -v_2 f_i(\omega t) + v_2 f_r(\omega t) = -v_1 f_t(\omega t)$

From (1.) and (2.):

$$f_r(\omega t) = \frac{v_2 - v_1}{v_2 + v_1} f_i(\omega t) \qquad R = \frac{v_2 - v_1}{v_2 + v_1}$$
$$f_t(\omega t) = \frac{2v_2}{v_1 + v_2} f_i(\omega t) \qquad T = \frac{2v_2}{v_1 + v_2}$$

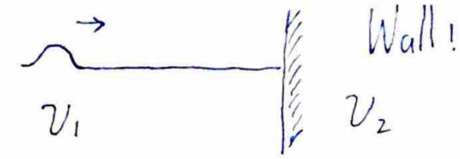
In this example:

$$v_2 = \frac{v_1}{2} \quad \Rightarrow \quad R = -\frac{1}{3} \quad T = \frac{2}{3}$$

The wave length changed, but the frequency did not. Two things we have learned:

1. The amplitude of the transmitted and reflected wave is determined by the properties of the two systems. “Impedance” in this case is $Z = T/V$
2. Wavelength changes: $k_1 \propto v_1^{-1}$

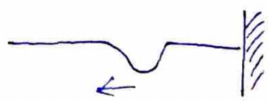
Consider two extreme cases:
The first is the string attached to a wall. In a sense, the “ρ_L” of the wall is very big, infinite in fact. Therefore $v_2 \to 0$ and $R = -1 \quad T = 0$. The amplitude changes sign but not magnitude, and there is no transmitted wave.
In the second case, there is air on the other side. The “ρ_L” of air is 0, therefore $v_2 \to \infty$ and $R = 1 \quad T = 2$

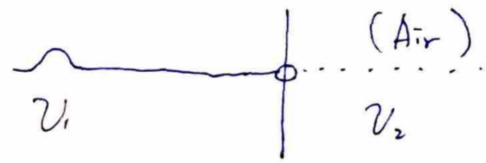
(1) 

$\rho_{L,2}$ of Wall is BIG!

$\Rightarrow \quad v_2 \to 0$

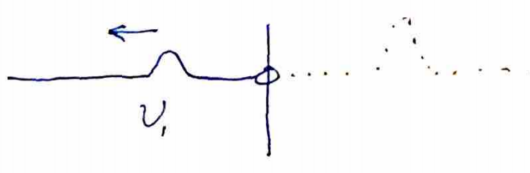
$\Rightarrow \quad R = -1 \quad , \quad T = 0$



(2) 

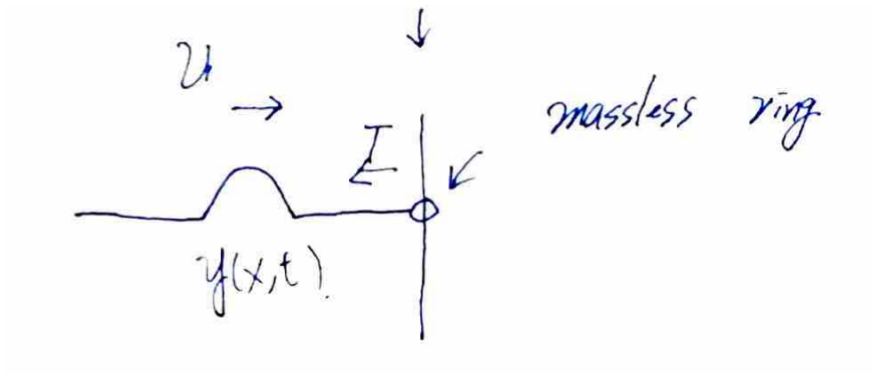
$\rho_{L,2} \longrightarrow 0 \quad \Rightarrow \quad v_2 \to \infty$

$\Rightarrow \quad R = 1 \quad , \quad T = 2$

$\Rightarrow$ 

More examples:

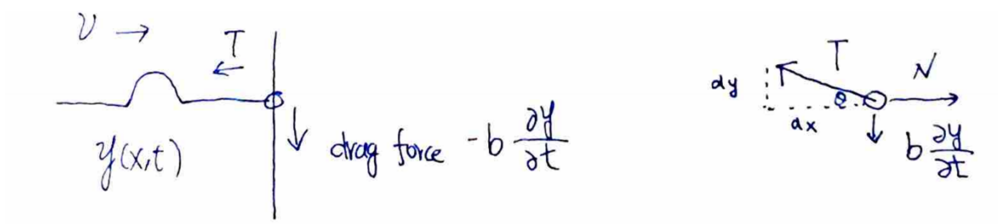
Example 2 (driven massless ring):



Boundary conditions:

1. $y\big|_{x=0} = P(t)$ a driving force.
2. Tension force cancels the normal force: $\Rightarrow \quad -T\frac{\partial y}{\partial x} = 0$

<u>Example 3:</u>



1. $y\big|_{x=0} = P(t)$
2. $-T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t} = 0$ (Where $\sin\theta \approx \frac{\partial y}{\partial x}$)