Formula Sheet Exam 2

Springs and masses:

$$m\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A\cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2\Gamma^2}}$$

and

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:
For $\Gamma = 0$ (undamped system):

$$x(t) = R\cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = Re^{-\frac{\Gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}\; t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.
For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t)e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.
For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.

Coupled oscillators

$$F_j = -\sum_{k=1}^{n} K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices $\mathcal{M}$, $\mathcal{K}$, $\mathcal{I}$ are $n \times n$, vectors $\mathcal{X}$, $\mathcal{Z}$ are $n \times 1$.

$$\frac{d^2}{dt^2}\mathcal{X}(t) = -\mathcal{M}^{-1}\mathcal{K}\mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A}e^{-i\omega t}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I})\mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$det(\mathcal{M}^{-1}\mathcal{K} - \omega^2\mathcal{I}) = 0$$

For $n = 2$

$$det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11}M_{22} - M_{12}M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d)e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1}\mathcal{K} - \omega_d^2\mathcal{I})\mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudies for the first ($c_1(\omega_d)$) and second ($c_2(\omega_d)$) objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues (β) and eigenvectors ($\mathcal{A}$) of this 2×2 $\mathcal{S}$ matrix:

(1) $\beta = -1$, $\mathcal{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) $\beta = 1$, $\mathcal{A} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β, the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of jth(0th) object in the system.

Consider an one dimentional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node} : \sum_i I_i = 0 \ \text{Loop} : \sum_i \Delta V_i = 0$$

$$\text{Capacitors} : \Delta V = \frac{Q}{C} \quad \text{Inductors} : \Delta V = -L\frac{dI}{dt} \quad \text{Current} : I = \frac{dQ}{dt}$$

Trigonometric equalities:

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

Integrals involving sin and cos:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$
$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$
$$\frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$
$$\int x\sin(x)dx = \sin(x) - x\cos(x) + C$$
$$\int x\cos(x)dx = \cos(x) + x\sin(x) + C$$

Maxwell Equations in vacuum

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \ ; \ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \ ; \ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\epsilon_0\frac{\partial E_z}{\partial t} \ ; \ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \ ; \ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0\epsilon_0\frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \ ; \ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

Wave equation for EM fields in vacuum

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \text{ where } \ i = x, y, z$$

$$\frac{\partial^2 B_i}{\partial x^2} + \frac{\partial^2 B_i}{\partial y^2} + \frac{\partial^2 B_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 B_i}{\partial t^2} \text{ where } \ i = x, y, z$$

For EM plane waves in vacuum:

$$\vec{B}(\vec{r}, t) = \frac{1}{c}\hat{k} \times \vec{E}(\vec{r}, t)$$

$$\vec{E}(\vec{r}, t) = c\vec{B}(\vec{r}, t) \times \hat{k}$$

Linear energy density in a string with tension T and mass density ρ_L

$$\frac{dK}{dx} = \frac{1}{2}\rho_L\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

EM energy per unit volume and Poynting vector:

$$U_E = \frac{1}{2}\epsilon_0\vec{E}^2 \qquad U_B = \frac{1}{2\mu_0}\vec{B}^2 \qquad \vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

Transmission and reflection

$$R = \frac{z_1 - z_2}{z_2 + z_1} \qquad\qquad T = \frac{2z_1}{z_2 + z_1}$$

Phase velocity and impedance:

$$v = \sqrt{\frac{T}{\rho_L}} \qquad\qquad Z = \sqrt{T\rho_L} \ \text{ (string)}$$

$$v = \sqrt{\frac{1}{LC}} \qquad\qquad Z = \sqrt{\frac{L}{C}} \ \text{ (transmission line)}$$

Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Fourier transform

$$f(t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t}$$

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt = \delta(\omega - \omega')$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$