BARTON ZWIEBACH: Hydrogen atom is the beginning of our analysis. It still won't solve differential equations, but we will now two particles, a proton, whose coordinates are going to be coordinate of the proton, subbing Xp for the proton, and momentum of the proton. And there's an electron. And there's the coordinates, the three coordinates of the electron, and the three components of the momenta of the electron.

And these are your canonical variables. This means that the components of this object satisfy the standard commutation relations. That is-- I have to write it like the following. They considered the coordinates of the proton, the i-th component. And the momentum of the proton, the j-th component, that's equal to ih bar delta ij.

You see, we used to code for its x, y, and z, and momenta Px, Py, Pz. You could've called it X1, X2, X3, momenta P1, P2, P3. And in that way, you can use a parameter delta over here. So, these are the commutation relations of X's and P's, but X and P for a proton. So, the proton has its X, has its P, and it behaves like an X and P that we've studied.

The electron has its X, its P, and also behaves the same way as coordinates P of electron j equal ih bar delta ij. You see, this is because x operator, y operator, z operator-- you can call them xi operators with 1, 2, 3. The Px, Py, Pz operators are better called P sub i, with i running from 1, 2, and 3, and x being the first, y the second, z being the last.

Then the fact that the x just fails to commute with Px and y fails to commute with Py, and z fails to commute with Pz is xi pj equal ih bar delta ij. And this is what I'm saying here with this notation because there are too many subscripts. There's a P for proton, but there's an i for the first, second and third component. And there's the momentum of proton, which has 1, 2, 3 components. So, those are our dynamical variables.

And then when you try to solve this, you have the issue of wave functions. What happens to the wave functions? Should I invent a wave function for the electron and a wave function for the proton? No. That's not good. You shouldn't a wave function for everybody, which is a wave function that depends on the coordinates X of the electron. And if you had just an electron, you would have a wave function that depends on X of the electron and a wave function that depends on the coordinates of the proton.

And how would you normalize it? Well, this thing times d cubed Xe times d cubed Xp is the probability, dP to find the electron around the little cube about the point, Xe, and the proton around the little cube around Xp. And then if you want to see what is it probability to find the electron at some point regardless of the proton, you integrate over the whole proton.

And then you should just have a wave function. So, if you integrate this psi squared-- oh, I'm sorry, psi is squared. If you this psi squared over the proton, you're left with some probability density for the electron. So, this is the good thing about having more and more particles. You still have just one wave function and one Schrodinger equation. You have more particles, and it's just one wave function.

That's why people eventually talk about the wave function of the universe. I say, well, the whole universe has one wave function for every, for every molecule, for every elementary particle. There's just one wave function. That's nice about the Schrodinger equation.

And, OK, so we have this, and what is the Hamiltonian? The Hamiltonian is the kinetic energy associated to the proton plus the kinetic energy associated to the electron plus the potential energy, which just the depends on the distance between the electron and the proton.

And then and write the differential equation, the Schrodinger equation. And the P of the proton would be thought as ddx of the proton. And the P of the electron would be like ddx of the electron. And that differential equation would make sense.

So what is our problem showing that we have-- can think in some way as a potential or a Schrodinger equation that the most-- just some distance that separates the particles, and some equation for the rest. So, we have to change variables.

So what is the change of variables? I'm going to motivate some of that. Well, we have two pairs of canonical variables. We have the proton canonical variables, X and P, of the proton, and the electron canonical variables, X and P, of the electron. So how can we have different kind of variables that will lead to a more interesting thing?

We can imagine that maybe we should define a new X that is the difference for which the magnitude of that X will be what we call r. And that's true, but it's not so easy to do that from the beginning, so let's try to do what most people would do from experience and say, I know there's something simple about a system of doing the wrapping values, center of mass. If the center of mass moves with constant velocity, that's what it does always.

So, if classical notions of how you treat two-body systems can be used in a quantum setting, we should be able to define a new quantum coordinate associated to the center of mass, and a new quantum momentum associated to the center of mass.

And the center of mass momentum is always the sum of the momenta, so we will define at P, a capital P-- I hope that my P's are going to be always clear. They're not supposed to have before this one, the little bar below. This is the total, P, and it's going to be defined as the momentum of the proton plus the momentum of the electrons.

And now, I want to define an X that goes with this P. And by that, I mean that this X must have a commutation relation with this P that tells, yes, you are an X because X with P should be ih bar. So, whatever I define here, that should happen. Now, you could imagine that we usually do something like this-- we multiply each particle according to its mass. Pe.

So we're weighting the momenta associated to-- oh, I'm sorry. The positions, Xp, Xe, associated to the mass. And this, of course, would not have the right units. It doesn't have units of coordinate, so this is the center of mass. So, this will be center of mass quantum variables.

And in order to have units, you must divide by a mass, so it's not too unreasonable to put the sum of masses here. That's how you define it anyway in classical mechanics. And now, we must ask is X with P the i-th component of the j-th component. Is it really equal to ih bar delta ij?

And the answer, I would say, is yes because, here, let's do this one. You will have mp Xpi plus me Xei over mp plus m e. You see, once you do one, all the rest you should be able to do just without thinking.

P, on the other hand, is P of the momentums of j plus the momentum of the electron, so j. What is it equal to? And then you say, look, from Xp with momentums of P, yes, they give me ih bar delta ij, but there will be ih bar. The first factor there will be an extra mp over mp plus m e. So this contribution comes from the first term that only talks to the momentum of the proton. Doesn't talk to the momentum of the electron.

And the second term for the coordinate of the electron only talks to the momentum of the electron. And it does give you an ih bar delta ij, but this time will be with an m e over mp plus m e. And this factor is equal to 1. So, yes, we can box this. This is a pair of quantum mechanical canonical variables. They have the right commutation. They have the right units, the right commutator, the right everything.

So next time when you see this, you will say, X with P-- you say, Xp with P is 1. Xe with P is 1. mp plus-- 1. Yes, it works. You forget ih bar delta ij. This is more clear that it will work. Try it.

Next, we need the second pair of canonical variables, so-- well, actually I should use this one for this. So for the second pair, it's reasonable to use the X that we anticipated, so we'll have a relative coordinate, X. And I don't know how I call it-- X relative, or-- no, just little x. Little x will be defined as X of the electron. I think that's coordinates. X of the electron minus X of the proton. This is natural. We want a little x like that.

Already at this point, you could be paralyzed with fear. Something could have gone wrong at this moment. Imagine if this x doesn't commute with these guys. Then it's all a disaster because this should be electron and proton commute with each other. I should have written there, not only those are the commutation relations, everything else is 0. Any X of electron with a momentum of a proton is 0. That's why they're two independent pairs of canonical variables that we can treat. It better be that this is an independent pair of canonical variables, so it better be that this x and whatever P I'm going to define here commute with these guys.

And as far as this x, happily it works out because X's commute with any X's here, so this little x definitely commutes with the little x. But the fact that this commutes with P could have killed it, but it doesn't because of the minus sign. One of the commutators of the little x with the capital P would give for the electron an ih bar, and for the proton, a minus ih bar. And they cancel. So, all is good so far.

So, this is so far so good-- commutes with capital X and capital P. So now, we have here something, and we could put that number, alpha Pe minus beta P of the proton. And we all know what alpha and beta are. But now you can more or less be confident. I need that this momentum with this x give me 1, ih bar. So, that would put the condition on alpha and beta.

In fact, the condition that xi with P bar j give you ih bar delta ij, you can imagine what it is. It's that alpha plus beta is equal to 1. And the condition that this momentum commute with the center of mass position, which it can fail to do so-- what can it give? Well, W alpha of the P of the electron goes with m e, so it's something like alpha m e minus beta mp is equal to 0.

And please-- oh, this commutator should be 0. Please make sure you know how to do this.

You can get those conditions. I'm going maybe a little fast for you to just follow it up. And so, at this moment, we can solve for alpha and beta-- two equations. So alpha is equal to mp over me plus mp. And beta is equal to m e over m e plus mp.

And we have a pair of canonical variables as well here, therefore, the relative coordinate and the relative momentum, in some sense. It's useful to define two symbols. So, we use mass m e mp over mp plus mp. And the total mass, which is m e plus mp. In the case of a heavy proton, the reduced mass-- the proton is heavy compared with the electron. You can ignore the electron here, cancel them, e. And the reduced mass is called this. So, in terms of the reduced mass, alpha is equal to mu over m e, and beta is mu over mp. Those are unit-free constants.

So summarizing, the second pair of canonical variables are Xe minus-- Xe minus Xp. And P equals mu Pe over m e minus P sub p over mp.

All right, so-- so, the last step that we're going to follow, and after that, we'll get the Hamiltonian and stop there. And we'll discuss solutions next time. But finally, we have enough equations to solve for the momenta we have in the Hamiltonian in terms of the center of mass momenta and the relative momenta.

So, from equation star and double star here, you can solve for these momenta. So, P of the proton will be equal to mp over M, big momenta minus little momenta. And P of the electron is m e over capital M, big momenta plus little momenta.

Well, one more equation that we have to write, and that's the key to the simplicity of all what we've done. At this moment, we have a very good physical insight into what we've done, but we want to see if the math collaborates. And the good physical insight has been that center of mass motion should be independent of the relative motion, and of we have the right X.

So, the thing that we have to compute is the first two terms in the Hamiltonian, P of the proton squared over 2m of the proton plus p of the electron squared over 2 mass of the electron. So, I have 1 over 2m of the proton, and I have m of the proton over capital M, P minus little p, and plus 1 over 2m of the electron-- this is squared-- m of the electron over capital M, big P plus little p.

And the plus and minus are extremely reassuring because the cross terms that couple the two are going to vanish. You can see cross terms will have a minus sign. mp will be canceled. m e will cancel. They will cancel. So at the end of this little calculation that takes couple of lines, you get 1 over 2 capital M total center of mass of this squared plus 1 over 2 mu little momentum squared.

This is a kinetic operator. And then you say, great success, the kinetic energy now has separated into a center of mass contribution and a relative contribution. This will allow us to now with a little step, separate the total Schrodinger equation into center of mass motion and relative motion, and the relative motion will have [INAUDIBLE] potential. So we'll do the punchline next time.