MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

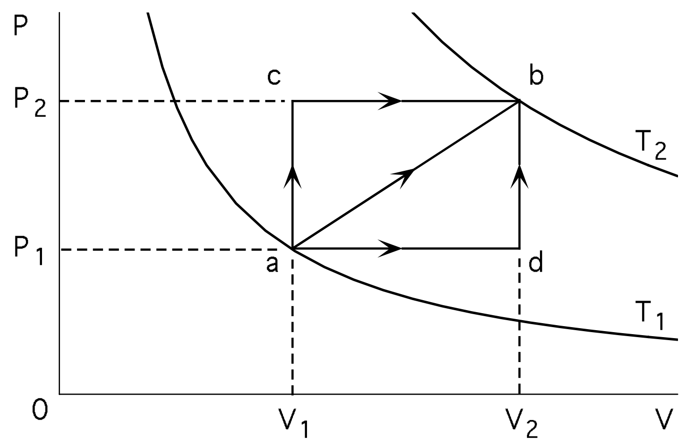
Problem Set #4

Due in hand-in box by 12:40 PM, Wednesday, March 6

Problem 1: Heat Capacity at Constant Pressure in a Simple Fluid

For a simple fluid show that $C_P = (\partial U/\partial T)_P + \alpha V P$. Since the thermal expansion coefficient α can be either positive or negative, C_P could be either less than or greater than $(\partial U/\partial T)_P$. [Hint: Use the first law to find an expression for $đQ$, then expand in terms of the variables T and P.]

Problem 2: Heat Supplied to a Gas



An ideal gas for which $C_V = \frac{5}{2}Nk$ is taken from point a to point b in the figure along three paths: acb, adb, and ab. Here $P_2 = 2P_1$ and $V_2 = 2V_1$. Assume that $(\partial U/\partial V)_T = 0$.

a) Compute the heat supplied to the gas (in terms of N, k, and T_1) in each of the three processes. [Hint: You may wish to find C_P first.]

b) What is the "heat capacity" of the gas for the process ab?

Problem 3: Thermodynamics of a Curie Law Paramagnet

Simple magnetic systems can be described by two independent variables. State variables of interest include the magnetic field H, the magnetization M, the temperature T, and the internal energy U. Four quantities that are often measured experimentally are

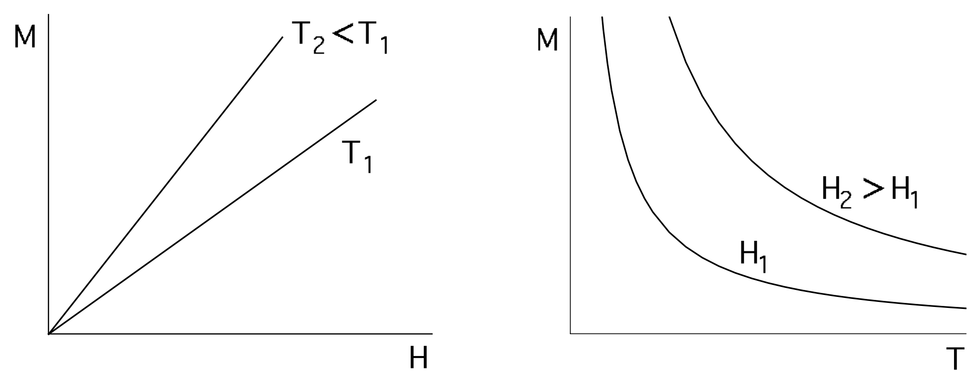
$$\chi_T \equiv \left(\frac{\partial M}{\partial H}\right)_T, \quad \text{the isothermal magnetic susceptibility,}$$

$$\left(\frac{\partial M}{\partial T}\right)_H, \quad \text{the temperature coefficient,}$$

$$C_M \equiv \left(\frac{đQ}{dT}\right)_M, \quad \text{the heat capacity at constant } M\text{, and}$$

$$C_H \equiv \left(\frac{đQ}{dT}\right)_H, \quad \text{the heat capacity at constant } H.$$

A particular example of a simple magnetic system is the Curie law paramagnet defined by an equation of state of the form $M = aH/T$ where a is a constant.



For such a system one can show that $(\partial U/\partial M)_T = 0$ and we shall assume that $C_M = bT$ where b is a constant.

a) Use T and M as independent variables and consider an arbitrary simple magnetic system (that is, not necessarily the Curie law paramagnet). Express C_M as a derivative of the internal energy. Find an expresson for $C_H - C_M$ in terms of a derivative of the internal energy, H, and the temperature coefficient. Write an expression for $dU(T, M)$ where the coefficients of the differentials dT and dM are expressed in terms of measured quantities and $H(T, M)$.

b) Find explicit expressions for $C_H(T, M)$ and $U(T, M)$ for the Curie law paramagnet. You may assume that $U(T = 0, M = 0) = 0$.

c) Consider again an arbitrary simple magnetic system, but now use H and M as the independent variables. Write an expression for $dU(H, M)$ where the coefficients of the differentials dH and dM are expressed in terms of the measured quantities and H. Is the coefficient of the dM term the same as in part a)?

d) Find explicit expressions for the coefficients in c) in the case of the Curie law paramagnet. You will need your result from b) for C_H. Convert the coefficients to functions of H and M, that is, eliminate T. Integrate $dU(H, M)$ to find $U(H, M)$. Compare your result with that found in b).

e) Using T and M as the independent variables, find the general constraint on an adiabatic change; that is, find $(\partial T/\partial M)_{\Delta Q=0}$ in terms of a derivative of the internal energy, $H(M, T)$, and $C_M(M, T)$.

f) Evaluate $(\partial T/\partial M)_{\Delta Q=0}$ for the Curie law paramagnet and integrate the result to find the equation of an adiabatic path in the T, M plane through the point T_0, M_0.

g) For the Curie law paramagnet, draw an isothermal path on a plot of M verses H. Pick a point on that path; show that the slope of an adiabatic path going through that point is less than the slope of the isothermal path.

Problem 4: Classical Magnetic Moments

Consider a system made up of N independent classical magnetic dipole moments located on fixed lattice sites. Each moment $\vec{\mu}_i$ has the same length μ, but is free to rotate in 3 dimensions. When a magnetic field of strength H is applied in the positive z direction, the energy of the i^{th} moment is given by $\epsilon_i = -m_i H$ where m_i is the z component of $\vec{\mu}_i$ (that is, $\vec{\mu}_i \cdot \hat{z} = m_i$).

The magnetization M and the total energy E are given by

$$M = \sum_{i=1}^{N} m_i \qquad\qquad E = \sum_{i=1}^{N} \epsilon_i = -MH$$

a) What are the physically allowed ranges of values associated with m_i, M, and E?

b) How many microscopic variables are necessary to completely specify the state of the system?

In a certain limiting case, the accessible volume in phase space for the microcanonical ensemble is given by

$$\Omega \approx (2\mu)^N \exp[-\frac{M^2}{\frac{2}{3}N\mu^2}].$$

c) Use the microcanonical ensemble to find the equation of state, M as a function of H and T.

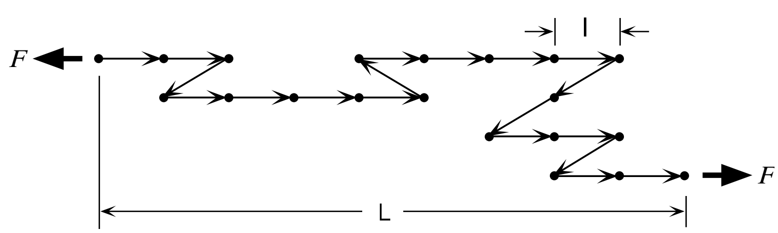
d) Is there some condition under which the solution to c) is unphysical for the system under consideration? Explain your answer. For what values of T is the expression for Ω a good approximation?

e) The probability density $p(M)$ for the z component of a single magnetic moment can be written as $p(m) = \Omega'/\Omega$ where Ω is given above. What is Ω'?

f) Find $p(m)$. [Note: For the limit which applies here, an expression for $p(m)$ including powers of m no higher than the first is adequate.] Sketch $p(m)$ and check its normalization.

g) Use $p(m)$ to computed $< m >$. Compare the result with that which one would expect.

Problem 5: A Strange Chain



A one dimensional chain is made up of N identical elements, each of length l. The angle between successive elements can be either 0^0 or 180^0, but there is no difference in internal energy between these two possibilities. For the sake of counting, one can think of each element as either pointing to the right (+) or to the left (-). Then one has

$$\begin{aligned} N &= n_+ + n_- \\ L &= l(n_+ - n_-) = l(2n_+ - N) \end{aligned}$$

a) Use the microcanonical ensemble to find the entropy as a function of N and n_+, $S(N, n_+)$.

b) Find an expression for the tension in the chain as a function of T, N, and n_+, $\mathcal{F}(T, N, n_+)$. Notice the strange fact that there is tension in the chain even though there is no energy required to reorient two neighboring elements! The "restoring force" in this problem is generated by entropy considerations alone. This is not simply an academic oddity, however. This system is used as a model for elastic polymers such as rubber.

c) Rearrange the expression from b) to give the length as a function of N, T, and $\mathcal{F}$.

d) Use the result for the high temperature behavior from c) to find an expression for the thermal expansion coefficient $\alpha \equiv L^{-1}(\partial L/\partial T)_{\mathcal{F}}$. Note the sign. Find a stout rubber band. Hang a weight from it so that its length is extended by about a factor of two. Now heat the rubber band (a hair drier works well here) and see if the weight goes up or down.

Problem 6: Classical Harmonic Oscillators

Consider a collection of N identical harmonic oscillators with negligible (but non-zero) interactions. In a microcanonical ensemble with energy E, the system is on a surface in phase space given by

$$\sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2} \right) = E.$$

a) Find the volume of phase space enclosed, $\Phi(E)$, as follows. Transform to new variables

$$\begin{aligned} x_i &= \frac{1}{\sqrt{2m}}\, p_i \qquad & 1 \le i \le N \\ x_i &= \sqrt{\frac{m\omega^2}{2}}\, q_{i-N} \qquad & N+1 \le i \le 2N \end{aligned}$$

Note that in terms of these variables the constant energy surface is a $2N$ dimensional sphere. Find its volume. Find the corresponding volume in p-q space.

b) Find the entropy S in terms of N and E.

c) Find T and express E in terms of N and T.

d) Find the joint probability density for the position coordinate q_i and the momentum coordinate p_i of *one* of the oscillators. Sketch $p(p_i, q_i)$.

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