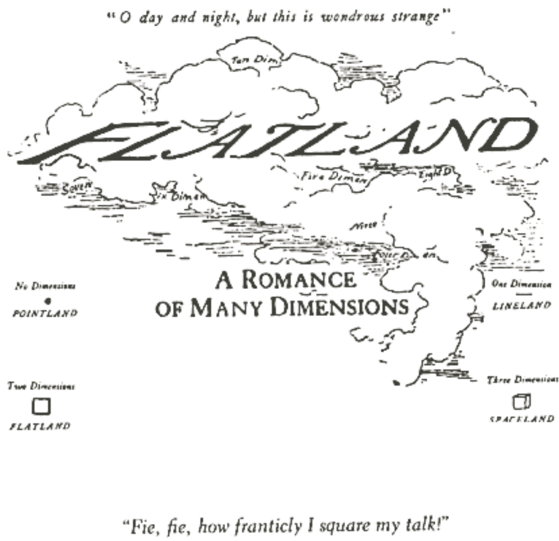
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I | Spring Term 2004

Exam #4

Problem 1 (35 points) Flatland



FLATLAND, Edwin A. Abbot, 1884

Consider world, perhaps Abbot's Flatland, where electromagnetic waves can only propagate in two dimensions, call them x and y. The electric field $\vec{E}$ must also be in the plane, but the magnetic field $\vec{B}$ is perpendicular to both the plane and the wavevector $\vec{k}$. The normal modes of the radiation field in a square box of side L with conducting walls are given by

$$\vec{E}_{k_x,k_y} = |E|\hat{1}\sin(k_x x)\sin(k_y y)\sin(\omega t + \phi)$$

where $\hat{1}$ is a unit vector in the direction of $\vec{E}$, k_x and k_y are determined by the need for $\vec{E}$ to go to zero at the walls, and $\omega = c|\vec{k}|$.

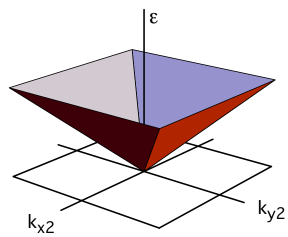
a) What are the allowed values of $\vec{k}$ in the box?

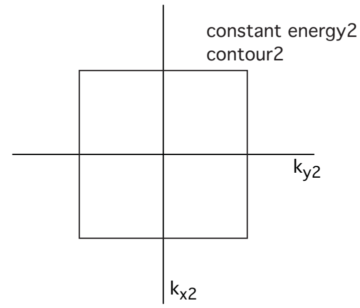
b) Find $D(\omega)$, the density of normal modes at frequency ω.

c) Find an expression for $u(\omega, T)$, the temperature dependent energy density (per unit area, per unit frequency interval) of thermal radiation in this world. Do not include contributions from the zero point energy in the field.

d) How is the Stefan-Boltzmann law changed in this world?

Problem 2 (35 points) Two-Dimensional Metal





We have studied electrons moving in a box in which the potential energy was zero. Alternatively one could consider electrons moving in a box containing a periodic potential – a simple model for the conduction electrons in a metal with a crystalline lattice. Under these conditions the single particle states can still be indexed by a wavevector $\vec{k}$; however, the energy of each state $\epsilon(\vec{k})$ need not be quadratic in $\vec{k}$ nor even isotropic in space.

The figure at the left above shows an approximation to the dispersion relation, $\epsilon(\vec{k})$, in a particular two-dimensional metal*. The energy has the form of an inverted square pyramid. It has four fold rotational symmetry. Along the k_x direction the energy is given by $\epsilon(k_x) = \gamma k_x$. The figure on the right shows a contour of constant energy on the k_x, k_y plane.

a) If one imposes periodic boundary conditions on the electron wavefunctions in a square sample of side L, what are the allowed values of the wavevector $\vec{k}$?

b) Find $D(\vec{k})$, the density of allowed wavevectors as a function of $\vec{k}$.

c) Find $D(\epsilon)$, the density of single particle states for the electrons as a function of their energy ϵ. Make a carefully labeled sketch of your result.

d) The metal contains N conduction electrons. Find the Fermi energy ϵ_F, the energy of the last single particle state occupied at $T = 0$.

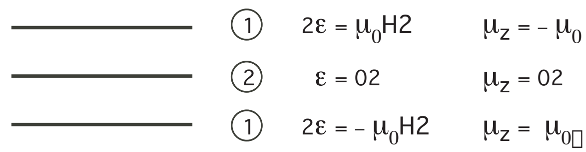
e) Find the total energy of the electrons at $T = 0$ in terms of N and ϵ_F.

f) Without doing any calculations, indicate how the electronic heat capacity depends on the temperature for temperatures $T << \epsilon_F / k_B$.

g) What is the surface tension $\mathcal{S}$ (the negative of the spreading pressure) of the electron gas at $T = 0$?

*Two dimensional planes of conduction electrons are not a fiction. They play an important role in semiconductor electronics and in high temperature superconductivity.

Problem 3 (30 points) Paramagnetic Ions



Certain impurity ions in a crystalline lattice interact with the neighboring atoms to create 4 states, 2 of which remain degenerate when a magnetic field H is applied along the z direction. The three resulting energy levels are shown above, along with their degeneracies, energies and magnetic moments.

a) Find the partition function for a single ion, $Z_1(T, H)$. You may wish to simplify the resulting expression using hyperbolic functions; see the information sheet for the properties of the hyperbolic functions.

b) Find the total energy $E(T, H) \equiv N < \epsilon >$ of N non-interacting ions in thermal equilibrium at temperature T.

c) Find the total magnetic moment (in the z direction) due to the N ions, $M(T, H)$.

You can check your answers to b) and c) by determining if they have the expected asymptotic behavior at low and high temperature.

Work in simple systems

Hydrostatic system	$-PdV$
Surface film	$\mathcal{S}dA$
Linear system	$\mathcal{F}dL$
Dielectric material	$\mathcal{E}d\mathcal{P}$
Magnetic material	HdM

Thermodynamic Potentials when work done on the system is $dW = Xdx$

Energy	E	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdX$
Enthalpy	$H = E - Xx$	$dH = TdS - xdX$

Results from hyperbolic trigonometry

$$\sinh(u) = (e^u - e^{-u})/2 \qquad \cosh(u) = (e^u + e^{-u})/2$$

$$\tanh(u) = \sinh(u)/\cosh(u) \qquad \coth(u) = 1/\tanh(u)$$

$$\frac{d}{dx}(\sinh\ u) = (\cosh\ u)\frac{du}{dx} \qquad \frac{d}{dx}(\cosh\ u) = (\sinh\ u)\frac{du}{dx}$$

Limiting behavior of	as $u \to 0$	as $u \to \infty$
$\sinh(u)$	u	$e^u/2$
$\cosh(u)$	$1 + u^2/2$	$e^u/2$
$\tanh(u)$	u	1
$\coth(u)$	$1/u + \frac{1}{3}u$	1

Statistical Mechanics of a Quantum Harmonic Oscillator

$$\epsilon(n) = (n + \tfrac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \ldots$$

$$p(n) = e^{-(n+\frac{1}{2})\hbar\omega/kT}/Z(T)$$

$$Z(T) = e^{-\frac{1}{2}\hbar\omega/kT}(1 - e^{-\hbar\omega/kT})^{-1}$$

$$< \epsilon(n) >= \tfrac{1}{2}\hbar\omega + \hbar\omega(e^{\hbar\omega/kT} - 1)^{-1}$$

Radiation laws

Kirchoff's law: $e(\omega, T)/\alpha(\omega, T) = \frac{1}{4}c\,u(\omega, T)$ for all materials where $e(\omega, T)$ is the emissive power and $\alpha(\omega, T)$ the absorptivity of the material and $u(\omega, T)$ is the universal blackbody energy density function.

Stefan-Boltzmann law: $e(T) = \sigma T^4$ for a blackbody where $e(T)$ is the emissive power integrated over all frequencies. ($\sigma = 56.9 \times 10^{-9}$ watt-m^{-2}K^{-4})

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