Quantum Physics II (8.05) Fall 2013
Assignment 4

Massachusetts Institute of Technology
Physics Department
September 27, 2013

Due October 4, 2013
3:00 pm

Problem Set 4

1. **Identitites for commutators** (Based on Griffiths Prob.3.13) [10 points]

   In the following problem A, B, and C are linear operators. So are q and p.

   (a) Prove the following commutator identity:

   $$[A, BC] = [A, B]\, C + B\, [A, C]\ .$$

   This is the derivation property of the commutator: the commutator with A, that is the object $[A, \cdot]$, acts like a derivative on the product BC. In the result the commutator is first taken with B and then taken with C while the operator that stays untouched is positioned at the expected place.

   (b) Prove the Jacobi identity:

   $$[[A, B]\, , C] + [[B, C]\, , A] + [[C, A]\, , B] = 0\, .$$

   (c) Using $[q, p] = i\hbar$ and the result of (a), show that

   $$[q^n, p\,] = i\hbar\, nq^{n-1}\, .$$

   (d) For any function $f(q)$ that can be expanded in a power series in q, use (c) to show

   $$[f(q), p] = i\hbar f'(q)\, .$$

   (e) On the space of position-dependent functions, the operator $f(x)$ acts multiplicatively and p acts as $\frac{\hbar}{i}\frac{\partial}{\partial x}$. Calculate $[f(x), p\,]$ by letting this operator act on an arbitrary wavefunction.

2. **Useful operator identities and translations** [10 points]

   Suppose that A and B are two operators that do not commute, $[A, B] \neq 0$.

   (a) Let t be a formal variable. Show that

   $$\frac{d}{dt} e^{t(A+B)} = (A+B)\, e^{t(A+B)} = e^{t(A+B)}\, (A+B)\ .$$

(b) Now suppose $[A, B] = c$, where c is a c-number (a complex number times the identity operator). Prove that

$$e^A B e^{-A} = B + c\ . \tag{1}$$

[Hint: Define an operator-valued function $F(t) \equiv e^{tA} B e^{-tA}$. What is $F(0)$? Derive a differential equation for $F(t)$ and integrate it.]

Comment: Equation (1) is a special case of the Hadamard lemma, to be considered below.

(c) Let a be a real number and $\hat{p}$ be the momentum operator. Show that the unitary **translation operator**

$$\hat{T}(a) \equiv e^{-ia\hat{p}/\hbar}$$

translates the position operator:

$$\hat{T}^\dagger(a)\,\hat{x}\,\hat{T}(a) = \hat{x} + a\ .$$

If a state $|\psi\rangle$ is described by the wave function $\langle x|\psi\rangle = \psi(x)$, show that the state $\hat{T}(a)|\psi\rangle$ is described by the wave function $\psi(x - a)$.

3. **Proof of the Hadamard lemma** [10 points]

Prove that for two operators A and B, we have

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots\ . \tag{1}$$

Define $f(t) \equiv e^{tA} B e^{-tA}$ and calculate the first few derivatives of $f(t)$ evaluated at $t = 0$. Then use Taylor expansions. Calculating explicitly the first three derivatives suffices to obtain (1).

Do things to all orders by finding the form of the $(n + 1)$-th term in the right-hand side of (1). To write the answer in a neat form we define the operator $\text{ad}\,A$ that acts on operators X to give operators via the commutator

$$\text{ad}\,A(X) \;\equiv\; [A, X]\,.$$

Confirm that with this notation, the complete version of equation (1) becomes

$$e^A B e^{-A} = e^{\text{ad}\,A}(B)\ .$$

4. **Special case of the Baker-Haussdorf Theorem** [10 points]

Consider two operators A and B, such that $[A, B] = cI$, where c is a complex number and I is the identity operator. You will prove here the following identity

$$e^{A+B} = e^B e^A e^{c/2} = e^A e^B e^{-c/2}\,. \tag{2}$$

For this purpose consider the operator valued function

$$G(t) \equiv e^{t(A+B)} e^{-tA}.$$

(a) Using operator properties and identities you derived previously show that

$$G^{-1}\frac{d}{dt}G(t) = B + ct\,. \tag{3}$$

(b) Note that (3) is equivalent to $\frac{d}{dt}G(t) = G(t)(B+ct)$. Verify that the solution to this equation is

$$G(t) \;=\; G(0)\, e^{tB}\, e^{\frac{1}{2}ct^2}\,. \tag{4}$$

(c) Consider $G(1)$ to prove the first equality in (2). Rename the operators to obtain the other equality.

Comment: The full Baker-Hausdorff formula is of the form

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\ldots}$$

and there is no simple closed form for general X and Y.

5. **Bras and kets.** [5 points]

Consider a three-dimensional Hilbert space with an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Using complex constants a and b define the kets

$$|\psi\rangle = a|1\rangle - b|2\rangle + a|3\rangle\ ; \qquad |\phi\rangle = b|1\rangle + a|2\rangle\,.$$

(a) Write down $\langle\psi|$ and $\langle\phi|$. Calculate $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$. Check that $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.

(b) Express $|\psi\rangle$ and $|\phi\rangle$ as column vectors in the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis and repeat (a).

(c) Let $A = |\phi\rangle\langle\psi|$. Find the 3×3 matrix that represents A in the given basis.

(d) Let $Q = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|$. Is Q hermitian? Give a simple argument (no computation) to show that Q has a zero eigenvalue.

6. **Shankar 1.8.8, p.43. Hermitian matrices and anticommutators** [5 points]

7. **Orthogonal projections and approximations** (based on Axler) [15 points]

Consider a vector space V with an inner-product and a subspace U of V that is spanned by rather simple vectors. (You can imagine this by taking V to be 3-dimensional space, and U some plane going through the origin). The question is: Given a vector $v \in V$ that is not in U, what is the vector in U that best approximates v? As we also have a norm, we can ask a more precisely question: What is the vector $u \in U$ for which $|v - u|$ is smallest. The answer is surprisingly simple: the vector u is given by $P_U v$, the orthogonal projection of v to U!

(a) Prove the above claim by showing that for any $u \in U$ one has

$$|v - u| \;\geq |v - P_U v|\,.$$

As an application consider the infinite dimensional vector space of real functions in the interval $x \in [-\pi, \pi]$. The inner product of two functions f and g on this interval is taken to be:

$$\langle f, g \rangle \; = \; \int_{-\pi}^{\pi} f(x) g(x) dx \, .$$

We will take U to be the a six-dimensional subspace of functions with a basis given by $(1, x, x^2, x^3, x^4, x^5)$.

In this problem please use an algebraic manipulator that does integrals!

(b) Use the Gram-Schmidt algorithm to find an orthonormal basis $(e_1, \ldots, e_6)$ for U.

(c) Consider approximating the functions $\sin x$ and $\cos x$ with the best possible representatives from U. Calculate exactly these two representatives and write them as polynomials in x with coefficients that depend on powers of π and other constants. Also write the polynomials using numerical coefficients with six significant digits.

(d) Do a plot for each of the functions ($\sin x$ and $\cos x$) where you show the original function, its best approximation in U calculated above, and the approximation in U that corresponds to the truncated Taylor expansion about $x = 0$.

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