Quantum Physics II (8.05) Fall 2013
Assignment 6

Massachusetts Institute of Technology
Physics Department
October 10, 2013

Due October 18, 2013
3:00 pm

Problem Set 6

1. **Exploring the time evolution of an overlap** [10 points]

   Consider a physical system governed by a *time-independent* Hamiltonian H. Let $\big|\Psi(0)\big\rangle$ denote the state of the system at $t = 0$ and $\big|\Psi(t)\big\rangle$ the state of the system at time $t \geq 0$. The state of the system satisfies the Schrödinger equation and is taken to be normalized. Consider now the overlap of the time-evolved state with the initial state, squared:

   $$\big|\langle\Psi(0)|\Psi(t)\rangle\big|^2$$

   (a) At time equal zero, the above equals one. Explain why it cannot ever exceed one. What is the value of the overlap if $\big|\Psi(0)\big\rangle$ is an energy eigenstate?

   (b) Calculate the overlap in a power series expansion valid for small t neglecting terms cubic and higher in t, namely, determine the terms represented by the dots in the equation

   $$\big|\langle\Psi(0)|\Psi(t)\rangle\big|^2 \;=\; \ldots \;+\mathcal{O}(t^3)$$

   Your answer will depend only on $t, \hbar$ and the uncertainty ΔH of the Hamiltonian!

2. **Exact inequalities for the time evolution of an overlap** [10 points]

   We proved an energy type uncertainty relation that read

   $$\Delta H\,\Delta Q \;\geq\; \frac{\hbar}{2}\Big|\frac{d\langle Q\rangle}{dt}\Big| \tag{1}$$

   for a Hermitian operator Q and a time-independent Hamiltonian H, showing also that ΔH is a constant in time.

   We will explore inequalities for the overlap $\big|\langle\Psi(0)|\Psi(t)\rangle\big|^2$. For this purpose and ease of manipulation we will write

   $$\cos^2\phi(t) \equiv \big|\langle\Psi(0)|\Psi(t)\rangle\big|^2\,, \tag{2}$$

   which makes clear that the right-hand side is never larger than one, and knowing the phase $\phi(t)$ is equivalent to knowing the overlap. At $t = 0$ we take $\phi(0) = 0$ and then as time evolves and the right-hand side becomes smaller than one we take ϕ to be in the interval

$$0 \leq \phi(t) \leq \frac{\pi}{2}\,. \tag{3}$$

This suffices, as it allows us to consider the possibility that the overlap becomes zero, when ϕ reaches the top value.

We denote the state of the system by $\big|\Psi(t)\big\rangle$ and take Q to be the projector to the state at $t = 0$

$$Q \;\equiv\; \big|\Psi(0)\big\rangle\,\langle\Psi(0)\,|\,. \tag{4}$$

(a) Use the uncertainty principle (1) to prove a surprising limit on the rate of change of the phase ϕ:

$$\left|\frac{d\phi}{dt}\right| \leq \frac{\Delta H}{\hbar}\,. \tag{5}$$

Since the ratio to the right is time independent, this is a very simple bound: the velocity of ϕ is limited by the energy uncertainty! Conclude that in a system governed by a time-independent Hamiltonian, the minimum time $\Delta t_{\perp}$ needed for any state with energy uncertainty ΔH to evolve into an orthogonal state satisfies the constraint

$$\Delta H \Delta t_{\perp} \;\geq\; \frac{h}{4}\,. \tag{6}$$

(b) Show that, as a consequence of (5) we have

$$\big|\langle\Psi(0)|\Psi(t)\rangle\big|^2 \;\geq\; \cos^2\Big(\frac{\Delta H t}{\hbar}\Big)\,, \quad \text{for} \;\; t \leq \frac{\pi\hbar}{2\Delta H}\,. \tag{7}$$

3. **Saturating a time-energy uncertainty relation** [10 points]

Consider a spin-1/2 particle in a magnetic field of magnitude B that points in the z direction. The Hamiltonian for such a system is

$$H \;=\; -\gamma B \hat{S}_z\,,$$

where γ is the constant that relates the spin to the magnetic moment of the particle. At time equal to zero the state of the particle $\big|\Psi(0)\big\rangle$ is such that the spin points along the positive x-axis.

(a) Calculate the time evolved state $\big|\Psi(t)\big\rangle$, and write your answer in terms of superposition of eigenstates of $\hat{S}_z$. Using your result describe in words how the direction of the spin of the particle changes in time.

(b) Show that in this example the uncertainty inequality $\Delta H \Delta t_{\perp} \;\geq\; \frac{h}{4}$ is saturated. Here $\Delta t_{\perp}$ is the time the spin takes to go into a configuration orthogonal to the original one.

4. **Sum rules and the quantum virial theorem** [10 points]

Consider the Hamiltonian

$$\hat{H} \;=\; \frac{1}{2m}\hat{p}^2 + V(\hat{x})\,,$$

for a one-dimensional quantum system with a discrete set of eigenfunctions:

$$H\big|a\big\rangle = E_a\big|a\big\rangle\,.$$

By evaluating matrix elements of suitable commutators, derive the following relations:

(a)

$$\sum_{a'} \left|\langle a|\hat{x}|a'\rangle\right|^2 (E_{a'} - E_a) \;=\; \frac{\hbar^2}{2m}\,.$$

This is known as the Thomas-Reiche-Kuhn sum rule.
Hint: Consider $[[\hat{x},\hat{H}],\hat{x}]$.

(b)

$$\langle a|\hat{p}|a'\rangle \;=\; \frac{im}{\hbar}(E_a - E_{a'})\langle a|\hat{x}|a'\rangle\,.$$

Hint: Consider $[\hat{H},\hat{x}]$.
Hence, show that

$$\sum_{a'} \left|\langle a|\hat{x}|a'\rangle\right|^2 (E_a - E_{a'})^2 \;=\; \frac{\hbar^2}{m^2}\langle a|p^2|a\rangle\,.$$

This is another energy-weighted sum rule.

(c)

$$\langle a|\frac{\hat{p}^2}{2m}|a\rangle = \frac{1}{2}\langle a|\hat{x}\partial_{\hat{x}}V(\hat{x})|a\rangle\,.$$

Hint: Consider $[\hat{x}\hat{p},\hat{H}]$.
This is the quantum mechanical virial theorem, which is usually stated as

$$2\langle T\rangle \;=\; \left\langle x\frac{dV}{dx}\right\rangle,$$

where T denotes kinetic energy and the expectation value is for a stationary state. For the case $V(x) = \alpha x^n$, write the resulting relation between expectation values of the kinetic and potential energy.

5. **Exercises on the one-dimensional harmonic oscillator** [10 points]

(a) We showed that the ground energy eigenstate $|0\rangle$ is the unique state annihilated by the lowering operator $\hat{a}$. Show algebraically that the excited states of the oscillator are non-degenerate, by showing that a degeneracy would imply a degeneracy of the ground state.

(b) Using $[a, a^\dagger] = 1$, prove that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$.

(c) Using the basis $\{|n\rangle\}$ find the matrix representation for the operators $\hat{a}, \hat{a}^\dagger, \hat{x}, \hat{p}$, $\hat{x}^2$, $\hat{p}^2$, and the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$. Do this by giving general formulae for the matrix elements $\mathcal{O}_{mn}$ of each operator $\mathcal{O}$. Write explicitly the corresponding four by four matrix truncations using $m, n = 0, 1, 2, 3$.

(d) Use the four by four matrices for $\hat{x}$ and $\hat{p}$ to compute $[\hat{x}, \hat{p}]$. Do you get the matrix $i\hbar I$? Explain.

(e) From your earlier result above you must have found that

$$\langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right), \quad \langle n|\hat{p}^2|n\rangle = m\hbar\omega\left(n + \frac{1}{2}\right)$$

Find the uncertainties Δx and Δp in the state $|n\rangle$. Is the product of uncertainties saturated? How is Δx related to the maximal excursion of the classical oscillator with the same energy (and same m, ω)?

6. **Asymmetric Two Dimensional Oscillator** [15 points]

Suppose a particle of mass m is free to move in the (x, y) plane subject to a harmonic potential centered at the origin. But suppose the restoring force in the x and y directions are different. The Hamiltonian for this system is

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\omega_x^2\,\hat{x}^2 + \frac{1}{2}m\omega_y^2\,\hat{y}^2 \tag{1}$$

where $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar$, and all other commutators between $\hat{x}$, $\hat{y}$, $\hat{p}_x$ and $\hat{p}_y$ are zero.

(a) Introduce lowering and raising operators $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_x^\dagger$ and $\hat{a}_y^\dagger$ as well as $\hat{N}_x = \hat{a}_x^\dagger\hat{a}_x$ and $\hat{N}_y = \hat{a}_y^\dagger\hat{a}_y$. What is $\hat{H}$ in terms of these operators? Find expressions for the energy eigenstates and the energy eigenvalues.
[The analogous results for the one-dimensional oscillator were $|n\rangle = \frac{1}{\sqrt{n!}}[\hat{a}^\dagger]^n|0\rangle$ and $E_n = \hbar\omega(n + \frac{1}{2})$. Here, you will want to define an n_x and n_y.]

(b) Plot an energy level diagram for this system. Let's assume, just to clarify the pictures, that $\omega_x \approx \omega_y$, and to be definite take $\omega_x > \omega_y$. Include at least the first three groups of states. Indicate their values of n_x and n_y.

Now define new operators,

$$\begin{aligned} \hat{N} &= \hat{N}_x + \hat{N}_y \\ \hat{n} &= \hat{N}_x - \hat{N}_y \end{aligned}$$

and notice that they commute with $\hat{H}$. The energy eigenstates can therefore be labelled by n and N, the eigenvalues of $\hat{n}$ and $\hat{N}$.

(c) What is $E_{N,n}$?. Redraw the energy level diagram and label the states with the quantum numbers n and N. Use your pictures to decide which of the following are complete sets of commuting observables: $\{\hat{N}\}$, $\{\hat{N}, \hat{n}\}$, $\{\hat{N}_x, \hat{N}_y\}$, and $\{\hat{H}\}$. How do your answers change if you take $\omega_x = \omega_y$? How do your answers change if ω_x/ω_y is equal to a rational number?

From now on we let $\omega_x = \omega_y = \omega$, and define the angular momentum operator

$$\hat{\ell} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \ . \tag{2}$$

(d) Write $\hat{\ell}$ in terms of the operators $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_x^\dagger$ and $\hat{a}_y^\dagger$. Show that $\hat{\ell}$ commutes with $\hat{H}$ and that therefore both can be simultaneously diagonalized.

(e) Consider the degenerate subspace consisting of all the energy eigenstates that have the N^{th} energy eigenvalue. Find a basis for this subspace such that the basis vectors are eigenstates of $\hat{\ell}$. Classify these basis states by their angular momentum eigenvalues, and show that $\hat{H}$ and $\hat{\ell}$ together constitute a complete set of commuting observables for the entire Hilbert space.

Hint: Define

$$\hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y), \quad \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y), \quad \hat{N}_L = \hat{a}_L^\dagger \hat{a}_L, \quad \hat{N}_R = \hat{a}_R^\dagger \hat{a}_R,$$

and express $\hat{H}$ and $\hat{\ell}$ in terms of $\hat{N}_L$ and $\hat{N}_R$.

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