BARTON ZWIEBACH: [INAUDIBLE] of today's lecture is coherent states of the harmonic oscillator. So let me begin by telling you about some things we've learned in the last lecture, and here they are. We learned how to calculate the so-called Heisenberg operators. Remember, if you have a Schrodinger operator, you subject it to this transformation with a unitary operator. That creates time evolution and that gives you the Heisenberg operator.

We learned things about Heisenberg expectation values. If the Hamiltonian is time independent, h, time independent, the formula is quite simple and gives you the Heisenberg operator at the later time. So we did this. We found, in fact, Heisenberg operators satisfy equations of motion.

And we calculated the Heisenberg operators for the harmonic oscillator. That was our main achievement last time, a formula for the time development of the x and p operators in the Heisenberg picture. And that really contains all the information of the dynamics, as you will see today, when we will be using this stuff.

Now, I suggested that you read-- and you may do it later. There's no need that you've done it for today-- the information on the time development of the creation and annihilation operators. You see, the a and a dagger are different inverses of x and p, are linear combinations. So the a and the a dagger operators also can be further Schrodinger operators that have no time dependence. And suddenly, if you go to the Heisenberg picture, the creation and annihilation operators become time dependent operators.

So that's in the notes. You can read about it. So we define the time dependent operator, a hat, to be the Heisenberg version of a hat. And you're supposed to do a calculation and try it or read it, and the answer is very nice, simply a phase dependence. The a is a at time equals 0, the Schrodinger 1 times e to the minus i omega t. Then a dagger is just what you would expect, the dagger of this, which the face has an opposite sign and a becomes a dagger.

Finally, if you substitute this a and a daggers in this formula. For example, you could say x Heisenberg is a Heisenberg plus a dagger Heisenberg. And you substitute those Heisenberg values there, you will obtain this. Same for the momentum. If you put Heisenberg, Heisenberg, Heisenberg, remember, if you have an equality of Schrodinger operators, it also holds when you put Heisenberg in every operator.

And therefore, if you put the Heisenberg a, a, and use those values, you will recover this equation. So in a sense, these equations are equivalent to these ones. And that's basically our situation. This is what we've learned so far, and our goal today is to apply this to understand coherent states of the harmonic oscillator.

Now, why do we want to understand coherent states of the harmonic oscillator? You want to understand coherent states because the energy eigenstates are extraordinarily quantum. The energy eigenstates of the harmonic oscillator don't look at all-- and you've seen the expectation value of the position. It's time independent. It just doesn't change. Expectation value of any operator in a stationary state is a constant. It just doesn't change.

So you have any eigenstate, any energy eigenstate of the harmonic oscillator, you ask, what is the position of this particle doing? Nothing. What is the momentum of this particle doing? Nothing. So nevertheless, of course, it's an interesting state, but we want to construct quantum mechanical states that behave a little like the classical states we're accustomed to. And that's what coherent states do.

We'll have an application of coherent states to light, photons, coherent photons. What are they? We'll see it later this week. So that's the reason we want to understand coherent states, because we want some states that in some ways behave classically, or close to classically. So they have many applications, these states, and you will see some of them in this lecture. I'm going to try to keep this blackboard there, untouched, so that we can refer to these equations.

So our first step is considering translation operators. So let's consider the unitary translation operator. So translation operators. So this translation operator that I will write as T sub x0 will be defined to be the exponential of e to the minus i p hat x0 over h bar.

You have seen such operators before. We've seen a lot of them in the homework. So first of all, why is it unitary? well, it's unitary because x0 is supposed to be a real number. p is Hermitian. Therefore, this with the i is anti-Hermitian, and an exponential of anti-Hermitian operator is unitary.

Now, it has, actually, a very simple property. The multiplication of two of those operators is what? Well, you have an exponential, e to the minus ipx0, and an exponential followed, e to the minus ipy0. Now, if you're well trained in 805, you should get a little nervous for a second because you don't know, can I treat it easily? And then you relax and say, yes, these two operators, whatsoever the numbers here, this with another one with a y0 would commute. Therefore, they can be put together in the exponential, and this is T of x0 plus y0.

No combo Baker-Hausdorff needed here. It's just straightforward. So what is Tx0 dagger? T x0 dagger, if you take the dagger, you change this i for a minus i, so it's exactly the same as changing the sign of x0. So this is T of minus x0.

And by this identity, T of minus x0 with a T of x0 would be T of 0, which is the unit operator. So T of minus x0 is the inverse of T of x0, confirming that the operator is unitary. The inverse is the dagger. So I used here that this is the inverse because T minus x0 times T x0 is T of 0 is equal to 1. So I could mention here, T of 0 is equal to the unit operator.

So these are our translation operators, but you don't get the intuition of what they do unless you compute a little more. And a little more than you should compute is this. What is T x0 dagger x T x0? And what is T x0 dagger p T x0?

Now, why do we ask for these particular things? Why don't I ask, what is x hat multiplied by T x0? Why do I ask this? It is because an operator acting on an operator always does this. If you say an operator is acting on another operator, the first operator that is acting, you put it here with its inverse. It happens to be unitary, so you put the dagger, and you put the operator here. And this is the right thing to do. It has a simple answer and a simple interpretation, as we'll see now.

So what is T, this commutator, supposed to be? Well, you can probably imagine what this is. You've calculated it in homework, so I will not do it again. This is x plus x0. So you get the operator, x, plus x0 times the unit operator.

That was done before. And here, you get just p. Why? Because p hat is the only operator that exists in this translation thing, so p commutes with p. So these two operators commute and the T tagger hits the T, and it's equal to 1, so that's a simple thing.

So why is this reasonable? It's because of the following situation. If you have a state, psi, you can ask, for example, what is the expectation value of x in the state psi? And if this state represents a particle that is sitting somewhere here, roughly, the expectation value of x is basically that vector that tells you where the particle is.

So you could ask, then, what is the expectation value of x in the state T x0 psi? So you want to know, what does T x0 really do? Here, it seems to say something, takes the operator and displaces it, but that seems abstract.

If you ask this question, this seems more physical. You had a state, you act with an operator, it's another state. How does it look? Well, this expectation value would be the expectation value of x on T x0 psi, and the [? brau ?] would be psi T x0 dagger.

So actually, that expectation value builds precisely this combination, and that's why it's meaningful. And since you know what this is, this is psi x plus x0 psi. This is equal to the expectation value of x in the original state plus x0 times 1. So the expectation value of x in the new state, the x0 psi, is the expectation value of x in the old state plus x0.

So indeed, if this is x, you could do this for vectors, and here is x0. Well, the expectation value of x in the new state, the T x0 operator, took the state and moved it by a displacement x0 so that the new expectation value is the old one plus x0. So that's physically why these things are relevant.

A couple of other things you've shown in the homework, and you could retry doing them, is that T x0 on the x state, by this intuition, should be the x plus x0 state. It moves the state to the right. And if psi has a wave function, psi of x, T x0 of psi has a wave function, psi of x minus x0, since you know that psi of x minus x0 is the wave function translated by x0 to the right. The sign is always the opposite one. When you write psi of x minus x0, the function has been moved to the right x0. So this is our whole discussion and reminder of what the translation operators are.

So we've got our translation operator. Let's see how we can use it. And we'll use it to define the coherent states. So here comes the definition of what the coherent state is. It's a beginning definition, or a working definition, until we understand it enough that we can generalize it. By the time we finish the lecture, this definition will be generalized in a very nice way, in a very elegant way.

So coherent states. So here it goes. I'm going to take the vacuum state of the harmonic oscillator, the ground state of the harmonic oscillator, and simply displace it with a translation operator by x0. So this is going to be e to the minus i p hat x0 over h bar 0.

And I want a name for this state, and that's the worst part of it. There's no great name for it. I don't know if any notation is very good. If it's very good, it's cumbersome, so I'll write it like this. A little misleading. I'll put a tilde over the state. You could say it's a tilde over the x, but it really, morally speaking, is a tilde over the whole state. It means that this thing, you should read there's an x0 here used for the translation operator that appears here.

So that's the state, x tilde 0. Intuitively, you know what it is. You have the harmonic oscillator potential. Here is x. The ground state is some wave function like that. This state has been moved to position x0, and presumably some sort of wave function like that, because this translates the wave function.

So the ground state moves it up there to the right. That's what it is. That's a coherent state. And there's no time dependence here so far, so this is the state at some instant of time. The coherent state, maybe call it at time equals zero. Let's leave time frozen for a little while until we understand what this state does. Then we'll put the time back.

So a few remarks on this. x0 x0 is how much? Now, don't think these are position eigenstates. That's a possible mistake. That's not a position eigenstate. This is a coherent state. If these would be position eigenstate, you say delta of this minus that, but it's nothing to do with that.

Can you tell without doing any computation what is this number? How much should be? Yes?

AUDIENCE: 1.

BARTON ZWIEBACH: It should be 1. Why? Because it's a unitary operator acting on this thing, so it preserve length. So this should be equal to 0 0, should be 1. Very good. No need to do the computation. It's just 1.

Psi associated to this state is the ground state wave function at x minus x0. Where this refers to the wave function, x0 is psi 0 of x. So this is what I was saying here. The wave function has been translated to x0, the remark over there.

So these are our coherent states and we want to understand the first few basic things about them so we can do the following simple computations. So if I have to do the following, if I have to compute the expectation value of any operator, A, on a coherent state, I use the fact that I want to go back to the vacuum, so I put T x0 dagger A T x0 0. Because that way, I trace back to what the vacuum is doing. It's much easier to do that than to try to calculate something from scratch.

So for example, we have here that x0 x x0, well, you would replace it by T x T, T dagger x T, which you know is x hat plus x0. We calculated it a few seconds ago, top blackboard. And therefore, you got what is the expectation value of x on the ground state? x0, very good.

And therefore, we just got x0, which is what you would expect. The expectation value of x on the coherent state is x0. You're there. You've been displaced.

How about the momentum, x0 p hat x0? Well, p acted by the translation operator is unchanged. Therefore, we got 0 p 0, and again that 0, so this state still has no momentum. It represents a T equals 0, a state that is over here. And just by looking at it, it's just sitting there, has no momentum whatsoever.

Another question that is interesting, what is the expectation value of the Hamiltonian on the coherent state? Well, this should be, now you imagine in your head, T dagger H T. Now, H is p squared over 2m, and that p squared over 2m gets unchanged. p squared over 2m is not changed because T dagger and T does nothing to it, T dagger from the left, T.

Nevertheless. the Hamiltonian has a 1/2 m omega squared x hat, and x hat is changed by becoming x hat plus x0. Well, we don't want to compute too hard, do too much effort here. So first, we realize that here's the p squared over 2m and here's the m omega squared x hat squared, so that's the whole Hamiltonian.

So we got 0 H 0 plus the extra terms that come here. But what terms come here? There's a product of an x0 and an x between 0 and 0. x0 is a number, so you have an x between 0 and 0, and that's 0. So the cross product here won't contribute to the expectation value, so the last term that is there is 1/2 m is a number omega squared, x0 squared.

And what is the expectation value of the Hamiltonian on the vacuum? It's h omega over 2 plus 1/2 m omega squared, x0 squared. And you start seeing classical behavior. The expectation value of the energy at this point is a little quantum thing plus the whole cost of stretching something all the way to x0. 1/2 of k squared, k for the oscillator, x0 squared.

So the energy of this thing is quite reasonably approximated, if x0 is large enough, by the second term, and this is the cost of energy of having a particle of the potential. So it's behaving in a reasonable way. You can do a couple more little exercises that I'll put here as things for you to check.

Exercise. x0 tilde x squared x0 tilde. Just calculate. It's just useful to have. x0 squared plus h bar over 2m omega. And x0 tilde p squared x0 tilde is mh omega over 2. And finally, x0 tilde xp plus px x0 tilde is equal to 0.

Any questions? These are exercises for you to practice a little these expectation values. Questions on what we've done so far? Yes?

AUDIENCE: You said these coherent states is most significant only in the ground state, or is it also important to use them for [INAUDIBLE]?

BARTON ZWIEBACH: Well, we've defined the coherent state by taking the ground state and moving it, and these are particularly interesting. You could try to figure out what would happen if you would take an excited state and you move it. Things are a little more complicated.

PROFESSOR: And in a sense, they can all be understood in terms of what we do to the ground state. So we will not focus on them too much. In a sense, you will see when we generalize this how what we're doing is very special, in at least one simple way.

So we'll always focus on translating the grounds. Other questions? Yes.

AUDIENCE: Where does the term [INAUDIBLE] arise and why does it persevere [INAUDIBLE]

PROFESSOR: OK, here is the thing of the coherent state. Is this an energy eigenstate at this moment? What do you think? Is this an energy eigenstate-- this state over here?

No, it won't be an energy eigenstate. There's something funny about it. Energy eigenstates are always diffuse things. They never look like that.

So this is not an energy eigenstate, and you've done things with non-energy eigenstates. They change shape. As they evolve, they change shape.

What we will see very soon is that this state, if we let it go, it will start moving back and forth without changing shape. It's going to do an amazing thing. Energy eigenstates-- you're super-close to energy eigenstates.

You get something that changes in time and the shape changes, and you've even done problems like that. But this state is so exceptional that even as we let it go in time, it's going to change, but the shape is not going to spread out.

Do you remember when you considered a pulse in a free particle, how it disappears and stretches away? Well, in the harmonic oscillator, this has been so well prepared that this thing, as time goes by, will just move and oscillate like a particle. And it does so coherently. It doesn't change shape.

When we talk about light, coherent light is what you get from lasers. And so if you want understand lasers, you have to understand coherent states. OK, so this brings us there to time evolution. So let's do time evolution.

So what will happen? We'll have a state x0 goes to x0 comma t. So that's the notation. That's what we'll mean by the state at a later time.

And how are we going to explore this? Well, we're all set with our Heisenberg operator, there. We'll take expectation values of things to figure out how things look.

So what do we have here? We'll ask for X0 t, and we'll put the Schrodinger operator in between here-- X0 t, and this is what we'll call the expectation value of A as time goes by in the X0 0 state. This is what we call this.

But then, we have the time evolution. So this is equal to the original state, Heisenberg operator of A-- original state. And if you wish, you could then put the t operator-- as we have in the top blackboard to the right-- and reduce it even more. But we've computed a lot of this coherent state expectation value, so let's leave it like that.

So you could, if you wish, say this is equal to 0-- T X0 dagger A T X0 0. So you can ultimately reduce the expectation values of things on the vacuum. So OK, we're all set. Let's try to do one.

And the reason this is a nice calculation is that the time evolution of this state is a little complicated. We'll figure it out later, but it's easier to work with the time evolved state. So here it goes-- what is the expectation value of X as a function of time on the X0 state?

Well, it says here take the X0 state, and take the Heisenberg value of X. So we have it up there-- X hat cosine omega t plus b hat over M omega sine omega t X0. Forget about time evolution of the coherent states. We evolved the operator.

On the other hand, we have that the expectation value of p is 0 in the coherent state, and the expectation value of X is X0. So end of story-- calculation over-- X0 cosine of omega t. That's expectation value in time. This thing is oscillating classically. That's nice as can be. So classical behavior again, of a quantum state.

How about expectation value of p X0 of t? If it's oscillating, it better be moving, and it better have some momentum. So let's put the momentum operator here, the Heisenberg one. So we'll have p hat cosine omega t minus m omega x hat sine omega t X0 tilde. And this is 0, but X has X0 there, so minus m omega X0 sine of omega t, which is equal to m d dt of the expectation value of X.

Here it is-- expectation value of X. m d dt of that is minus m omega X0 sine omega t. That's what it should be. And this thing is really oscillating classically-- not only the exposition, but the momentum is doing that.

Now, the other thing that we can compute-- and we want to compute-- is the key thing. You have this state. We said it's coherent evolution.

So the ground state is this state that is a minimum uncertainty packet. It has a delta X uncertainty mix and a delta p. Their product saturates the uncertainty in equality.

And when we move the state X0, well, the delta X will be the same. The delta p will be the same, and it's that. But now as it starts to move, we want to see if the shape is kept the same. Maybe it fattens up, and shrinks down, and does things in the middle.

So the issue of coherency of this state is the issue whether the uncertainties remain the same. If the uncertainties remain the same, and they are saturated-- the product is saturating the inequality, you know that the shape has to be Gaussian, and it must be the same shape that is running around.

So what we need to compute is the uncertainty in X, for example. So how do delta X of t and delta p of p behave? That's our question.

And let's see how they do. Well, we have this computation-- actually, if you don't have the Heisenberg picture, it's kind of a nightmare. With the Heisenberg picture, it's a lot easier.

Delta x squared of t would be the expectation value of X0 t of X squared, X0 t, minus the expectation value of X0 t X, X0 t squared. I wrote what the definition of the uncertainty squared is. It's the expectation value of the operator squared, minus the square of the expectation value of the operator.

And of course, everything is going to turn Heisenberg immediately, so this thing-- maybe I can go one more line here-- would be X0 X Heisenberg squared of t, X0 minus-- this is simple-- this we've calculated. it's that expectation value at the top is the expectation value of X in time. It's that. So this is minus X0 squared cosine squared of omega t.

So what do we have to do? We have to focus on this term. So this term is equal to X0. And you have X Heisenberg squared, so let's do it-- X squared cosine squared omega t plus p hat squared over m squared w squared sine squared omega t plus 1 over mw cosine omega t sine omega t X p plus pX X0 tilde.

That shows that term, and I just squared that thing, but that I suggested a few exercises here. This is 0. In fact, it's 0 in the ground state as well, so this is 0. X squared gives you the top equation-- X0 squared plus h bar over 2 m omega cosine squared omega t-- plus p squared over m squared w squared, so p squared is m h omega over 2. And then you have m squared omega squared sine squared omega t. And that's this whole term.

And the thing that we're supposed to do is subtract this here. You see that the X0 squared cosine squared of omega t cancels here. So what do we get? h bar over 2 mw cosine squared omega t.

But this thing is also h bar over 2 mw sine squared omega t. So this whole thing, all the times have disappeared-- delta X squared-- the time dependence here has disappeared with that, and the cosine squared with sine squared have combined, and you get h bar over 2 m omega, which was-- this is, I'm sorry, of t. We work very hard to put the t there. We should leave it.

The uncertainty as a function of time has not changed. It is the original uncertainty of the ground state. So this is moving in a nice way.

You're supposed to compute now as well the uncertainty in p. I leave that as an exercise-- delta p squared of t equal m h bar omega over 2. So this is an exercise. Practice with coherent states.

It's worth doing it, I think. Actually there's going to be a problem in the homework set, in which you're going to ask to do most of these things, including things I'm doing here on the blackboard. So you will practice this.

So between these two, delta p, delta X-- delta X of t, delta p of t is, in fact, equal to h bar over 2. And this is a minimum uncertainty thing. And it behaves quite nicely.

All right, so the name coherent now should make sense. You've produced a quantum state that has about the energy of a state that you're familiar with, and it moves classically, and it doesn't change shape as it moves, so it moves coherent.

So our next task, therefore, will be to understand this in the energy basis. Because in the energy basis, it looks like a miracle. You've suddenly managed to produce a set of states of different energies, created the superposition, and suddenly, it moves in a nice way. Why does that happen?

So we need to understand the energy basis. And as we do that, we'll understand how to generalize the coherent states completely. So let's go on with that, and let's explore this in the energy basis.

So what do we have? We have the coherent state-- no need to put the time yet-- is e to the exponential of minus i p hat X0 over h bar. There is, as you've seen already, at length scale in the harmonic oscillator-- famous length scale, and we'll have an abbreviation for it. It's the length scale d0 squared h bar over m omega.

You can use the parameters of the harmonic oscillator-- h bar and m and omega-- to produce a length scale. And that length scale is d0. It's essentially the uncertainty in the position in the ground state, up to the square root of 2.

It's the way-- you want to construct a length-- there it is-- the only way you can construct a length. So I'm going to use that notation. So let me put what the p is into that formula, and simplify it.

So this is on the vacuum-- I'm sorry, I stopped half the way. So this is the exponential of X0 over square root of 2d a dagger minus a on the vacuum. Plug in the p, get the h bars, and you will see that d enters in that way. It's the way it has to enter, because this exponential should have no units, and therefore X0 over d0 has no units, and the a's and the a daggers have no units. So it couldn't be any way different like that.

The i also shouldn't be there, because this operator-- the i was there to make this anti-Hermitian. But this, with this real, is already anti-Hermitian. You see, you take the dagger. It becomes minus itself.

So this is anti-Hermitian. No need for an i-- in fact, an i would be wrong, so there's no i. And that's this.

Now, we want to figure out how this looks in the energy basis, so what are we going to do? We're going to have to do something with that exponential. We're going to have to reorder it. This is a job for Baker-Campbell-Hausdorff.

Which one? Well, this one-- e to the X plus Y is equal to e to X, e to the Y, e to the minus 1/2-- I don't know this by heart-- XY, and it stops there. If and only if X commutator with Y commutes with X, and commutes with Y.

There was a problem in the test that there was an operator with Xp plus pX acting on X, and after you commuted, you get X again, and you have to keep including terms.

So this stops here if XY commutes with X and Y. And why do I want this? Because I actually want to split the creation and the manipulation operators. I want them in separate exponentials. We have that energy eigenstates are creation operators on the vacuum, but here I have creation minus destruction. So if I expand the exponential, I'm going to get lots of creation and destruction, and I'm going to spend hours trying to sort it out.

If you did expand it, I bet you won't see it through so easily-- probably will take you forever, and it might not work out. So expanding an exponential is something that we should be reluctant to do.

On the other hand, this is a nice option, because then you think of this as e to the X0 over square root of 2d0 a dagger minus X0 over square root of 2d0 a. And I chose this analogy X with this, and Y with this. It's like Y is this thing minus that, and X is that.

You could have done it the other way around, but then you would run into trouble again. Why? Because I want a Y factor that has the manipulators to be to the right of the X factor that has the creation.

Why? Because if I have annihilators closer to the vacuum, that's good. Annihilators closer to the vacuum is what you really want, because if you have creators close to the vacuum, they create state, but then you have the manipulators, and you have to start working them out.

On the other hand, if the annihilators are close to the vacuum, they just kill the vacuum and you can forget them. So it's very important that you identify X with this, and Y with this whole thing.

So that this is e to the X0 over square root of 2D0 a dagger, e to the minus X0 over square root of 2d0 a. And now you're supposed to do the commutator of these two things. And the commutator is the commutator of an a dagger with an a, and that is 1. So this commutator is a number-- crucial, because if it wasn't a number, if it would be an a or an a dagger, it would not commute with a, X, and Y, and you have to include more terms.

So the fact that this commutator is a number allows you to use this formula. So now we'll put this factor here, minus 1/2. Then you have X with Y, and that's X0 minus X0 over square root of 2d squared, minus-- and this factor squared-- an a dagger with a, which is minus 1. So that's that whole operator.

So let's write it. The coherent state, therefore, X0 tilde, is equal to e to the X0 over square root of 2d0 a hat dagger, e to the minus X0 over square root of 2d0 a, and this factor that seems to be e to the minus 1/4 X0 squared over d0 squared. And here is this nice vacuum. Yes, factor is right.

So what is this? Well, this is a number, so I can pull it to the left. And here is the exponential of the annihilator operator. Now expand the exponential. It's 1. That survives, but the first term has an a-- kills it. The second term has an a squared-- kills it. Everything kills it.

This thing acting on the vacuum is just 1. That's why this is simple. So what have we got? In the state X0 tilde is e to the minus 1/4 X0 squared over d0 squared times e to the X0 over square root of 2d0 a dagger on the vacuum.

Well this is nice-- not quite energy eigenstates, but we're almost there. What is this? e to the minus 1/4, X0 squared over d0 squared. And now expand. This is the sum from n equals 1 to infinity, 1 over n factorial. X0 over square root of 2d0 to the n, a hat dagger to the n on the vacuum.

And what was the nth energy eigenstate? You probably remember. The nth energy eigenstate is a dagger to the n on the vacuum over square root of n factorial.

So we've got a little more than the square root of 2 n factorial. So maybe I'll do it here. We get e to the minus 1/4 X0 squared over d0 squared, sum from n equals 1 to infinity, 1 over square root of n factorial, X0 over the square root of 2 d0 to the n times the nth energy eigenstate. It's a little messy, but not so bad.

I think actually I won't need that anymore. Well no, I may. I will. So let's write it maybe again. Well, it's OK. Let's write it as follows-- Cn n. OK, so I got some cn's and n's. So this is a very precise superposition of energy eigenstates, very delicate superposition of energy eigenstates.

Let me write it in the following way-- cn squared. Why would I care about cn squared? cn squared is the probability to find the coherent state in the nth energy eigenstate. The amplitude to have it in the nth energy eigenstate is cn. So that probability to find it in the nth energy eigenstate is cn squared, is the probability 4x tilde 0 2b in the nth energy eigenstate.

That is what? Exponential of minus 1/2 x0 squared over d0 squared. And I have to square that. So I have 1 over n factorial. I have to square that coefficient there. So it's x0 squared over 2d0 squared-- that's nice, it's the same one here-- to the n.

So it's easier to think of this if you invent a new letter, lambda, to be x0 squared over 2d0 squared. Then, cn squared is equal to e to the minus lambda lambda to the n over n factorial. Yes.

AUDIENCE: Is that something that we should expect to be true for any time of [INAUDIBLE], or is that something that we [INAUDIBLE]?

PROFESSOR: Well, let me say it this way. In a second, it will become clear that this was almost necessary. I actually don't know very deeply why this is true. And I'm always a little puzzled and uncomfortable at this point in 805. So what is really strange about this is that this is the so-called Poisson distribution. So there's something about this energy eigenstate that their Poisson distributed in a coherent state.

So these are probabilities, as I claimed, to find an n. And indeed, let's check the sum of the cn squareds from n equals 1 to infinity. Let's see what it is. And you will see, you cannot tinker with this. This is e to the minus lambda the sum from n equals 1 to infinity lambda to the n over n factorial.

And that sum-- it's not from n equals 1. It's 0 to infinity, I'm sorry. Did I write once anywhere? Yeah, it should be 0. OK, this is 0. There is the ground state, so from n equals 0 to infinity. And this is e to the minus lambda e to the lambda, which is 1.

So yes, this is Poisson distributed. It's some sort of distribution like that. So if you have the n's, the cn's, Poisson distributions have to do with if you have a radioactive material. It has a lifetime. And you say, how many events should I-- the lifetime is five years. How many events should you expect to happen in a week? These are Poisson distributed.

So it's a Poisson distribution. It's a very nice thing. So let me just make one more remark about it. And it's quite something. So one question that you could ask is, what is the most probable n? That's a good question. You have a coherent state. So it's going to have the superposition of the vacuum, the first.

What is the most probable n, so the expectation value of n? Now, I'm thinking of it probabilistically. So I'm thinking this is a probability distribution. Then, I will show it for you that this is really computing what you want. But probabilistically, what is the expectation value of n? You should sum n times the probability that you get n.

So this is sum over ne to the minus lambda lambda to the n over n factorial. So you got an n there. And the way you get an n there-- well, the e to the minus lambda goes out. And the n can be reproduced by doing lambda d d lambda on the sum. Because lambda d d lambda on this sum brings down this n, puts back the lambda so it gives you the thing you had, and that's what it is.

So here, you get e to the minus lambda lambda d d lambda of e to the lambda. And that is lambda, OK? So the expectation value of the most sort of not the peak, but the expected value of n in this distribution, the level that you're going to be excited is basically lambda. So if x0 is 1,000 times bigger than d0, you've moved this thing 1,000 times the quantum uncertainty. Then, you're occupying most strongly the levels at 1 million. You get x0 over d0 controls which n is the most likely.

Indeed, look, this n-- suppose you would compute x0 tilde n hat x0 tilde. This is what you would think is an occupation number. This sounds a little hand wavy. But this is the number operator, the expected value of the number operator, in the coherent state.

But this is-- you have that the coherent state is this. So let's substitute that in there. So you get two sums over n and over m. And you would have cm star m N n cn. I've substituted x0 and x0 dagger. The c's in fact are real.

And then, the number operator on here is little n. And then, you get the Kronecker delta. So this is sum over n and m cmcn-- it's real. And then, you get n delta m, n. So this is in fact the sum over ncn squared. So what we wrote here, this is really the expectation value of the number operator.

And one can do more calculations here. A calculation that is particularly interesting to discover what these states look like is the uncertainty in the energy. So that's another sort of relevant measure. How big is the uncertainty in the energy? What are, basically, the delta E associated to the coherent state? How does it look like? Is it very sharp?

So it's a good question. And it's in the notes. I leave it for you to try to calculate it. Delta E in the coherent state x0, how much is it? And it turns out to be the following-- h omega x0 over square root of 2d. So actually, maybe this is a little surprising. But delta E over h omega is equal to x0 over d.

So actually, the energy uncertainty for a classical look in coherent state-- I'm sorry, I'm missing a square root of 2 there. So what is a classical looking coherent state? It's a state in which x0 is much bigger than the quantum d. So x0 is much bigger than d.

So in that case, this is a large number for a classical state-- "classical" state. But in that case, the uncertainty in delta E is really big compared to the spacing of the harmonic oscillator. So you have-- here is the ground state. Here is h omega. Here is the coherent state, maybe. And you have a lot of energy levels that are excited.

So if x0 over this is 1,000, well, at least 1,000 energy levels are excited. But you shouldn't fear that too much. Because at the same time, the expectation value of E over delta E-- the expectation of E is something we calculated at the beginning of the lecture. You have the oscillator displaced.

So this is roughly 1/2 m omega squared x0 squared. Throw away the ground state energy. That's supposed to be very little. Delta E is h bar omega x0 over square root of 2. And this is, again, the same ratio.

So yes, this state is very funny. It contains an uncertainty that measured in harmonic oscillator levels contains many levels. But still, the uncertainty is much smaller by the same amount than the average energy. So this state is a state of some almost definite energy, the uncertainty being much smaller. But even though it's much smaller, it still contains a lot of levels of the oscillator.

So that I think gives you a reasonable picture of this. So you're ready for a generalization. This is a time to generalize the coherent states and produce the set of coherent states that are most useful eventually, and most flexible. And we do them as follows.

We basically are inspired by this formula to write the following operator. And here, we change notation. This x0 was here. But now we'll introduce what is called the alpha coherent state. Most general coherent state is going to be obtained by acted with a unitary operator on the vacuum. So far so good-- D of alpha unitary.

But now generalize what you have there. Here, you put a minus a dagger minus a, because that was anti-Hermitian, and you put the real constant. Now this alpha will belong to the complex numbers. Quantum mechanics is all about complex numbers. You've got complex vector spaces, complex numbers. It's all over the place.

So how do we do this? We do this exponential of alpha a dagger. And I want it to be anti-Hermitian. So I should put minus alpha star a on the vacuum. This thing for alpha equals-- this real number reduces to that. But now, with alpha complex, it's a little more complicated operator. And it's more general. But it's still unitary. And it preserves a norm. And its most of what you want from these states.

So the first thing you do to figure out what this operator does is to calculate something that maybe you would not expect it to be simple. But it's worth doing. What is a acting on the alpha state? Well, I would have to do a acting on this exponential of alpha a dagger minus alpha star a on the vacuum.

Now, a kills the vacuum. So maybe you're accustomed already to the next step. I can replace that product by a commutator. Because the other ordering is 0. So this is equal to the commutator. Because the other term when a is on the other side is 0 anyway.

And now I have to compute the commutator of a with an exponential. Again, it's a little scary. But A with an exponential e to the B-- it's in the formula sheet, this Campbell-Baker-Hausdorff again-- is A, B e to the B if A, B commutes with B.

Well, this is A. This is B. A with B is-- A with B, the exponent-- just alpha times 1 because of this. So it's a number. So this is safe. So you get alpha times the same exponential. But the same exponential means the state alpha-- a little quick, isn't it?

OK, A with B, this factor, was alpha. And e to the B anyway on the vacuum is the state alpha. So there you go. You have achieved the impossible. You've diagonalized a non-Hermitian operator. This is not Hermitian, and you found its eigenvalues. How could that happen?

Well, it can happen. But then, all of the theorems that you like about Hermitian operators don't hold. So it's a fluke. This can be done. But then, states that correspond to different eigenvalues will not be orthogonal, and they will not form a complete set of states, and nothing will be quite as you may think it would be.

But still, it's quite remarkable that this can be done. So this characterizes the coherent state in a nice way. They're eigenstates of the destruction operator. And they're the most general exponentials of creation and annihilation operators acting on the vacuum.

Now, we knew that when alpha is real, it has to do with x0. So we've put a complex alpha. What will it do? A complex alpha, what it does is gives the original coherent state some momentum. Remember, the original state that we had was an x0. And how did it move? x0 cosine of omega t. So at time equals 0, it had 0 momentum. This creates a coherent state at x0, and it gives it a momentum controlled by the imaginary part of this thing.

In fact, we can do this as follows. You can ask, what is the expectation value of x in this state? Well, x is written here. It's d over square root of 2 alpha a plus a dagger alpha. And look, these are easy to compute. a gives an alpha, gives you alpha. a dagger and alpha, you don't know what it is. But a dagger on [? brau ?] alpha is alpha star.

So this one you know on the right. This one you know on the left. It gives you the over square root of 2 alpha plus alpha star. So it's square root of 2d real of alpha. So the real part of alpha is the expectation value of x.

So I'll go here. I'm almost done-- waiting for the punch line. Similarly, you can calculate the expectation value of the momentum. It will be alpha p alpha. And p is a minus a dagger. So you're going to get alpha star minus alpha, so the imaginary part.

So p is actually square root of 2 h bar over d imaginary part of alpha. So the physics is clear. Maybe the formulas are a little messy. But when you have an alpha, the real part of alpha is telling you where you're positioning the coherent state. The imaginary part of alpha is telling what kick are you giving to it. And you now have produced the most general coherent state.

So how do we describe that geometrically? We imagine it, the alpha plane. And here it is. The alpha plane, here is the vector, the complex number alpha that you've chosen maybe for your state, some particular complex value alpha. On the x-axis, the real part of alpha is the expectation value of x over square root of 2d.

The real part of alpha is the expectation value of x over square root of 2d.

And the imaginary part of alpha is the expectation value of p over square root of 2 h bar d. And there it is, your state at time equals 0. What is it going to do a little later? Well, that will be the last thing I want to calculate for you. It's a nice answer. And you should see it. It's going to take me two minutes.

And what is it? Well, alpha at time t, you have to evolve the state-- e to the minus iHt over h bar on the state, which is e to the alpha a dagger minus alpha star a e to the output and into the iHt over h bar, and an e to the minus iHt over h bar on the vacuum. So I put the one here. I evolve with this.

But I take the state and put this and that. This is simple. It's e to the minus iH omega over 2. That's the energy of the ground state. But what is this part? It's pretty much the Heisenberg operator. But the sign came out wrong.

Well, it didn't come out wrong. It's what it is. It just means that what I have to put here is the Heisenberg operator at minus t. Because I have t for minus t. So this is e to the alpha a Heisenberg at minus t dagger minus alpha star.

I'm sorry, I have too many parentheses here. That's it, much better-- minus alpha star a Heisenberg of minus t acting on this thing. And what is this? Well, we have the formula for the Heisenberg states here. So you've got e to the alpha H a dagger of minus t would be e to the minus I omega t a dagger.

And here, you have minus alpha star e to the i omega t a on e to the minus i omega h bar over 2 times the vacuum. And look what has happened. Alpha has become alpha times e to the minus i omega t. Because the star is here. It's minus the star one. So the only thing that has changed is that this state, alpha at time t, is e to the minus ih bar omega over 2. I'm sorry, I'm missing a t here.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, I dropped-- yeah, minus i omega t over 2, minus i omega t over 2, minus i omega t over 2, times the coherent state, the time independent coherent state of value e to the minus i omega t alpha. That's a new complex number.

That's what has happened. The number alpha has become e to the minus i omega t. Now, this is a face for the whole state multiplicative. It's irrelevant. So what has this alpha done? It has been rotated by e to the minus i omega t. So this at the time t is the state alpha t. Here is the state alpha. And it has rotated by omega t.

So the coherent state can be visualized as a complex number in this complex plane. This real part is the expectation value of x at time equals 0 whose imaginary part is the expectation value of the momentum at time equals 0. And how does it evolve? This state just rotates with frequency omega all along and forever. All right, that's it for today. See you on Wednesday.

[APPLAUSE]

PROFESSOR: Thank you, thank you.