PROFESSOR: So let's get started. So I'm going to lecture today, Professor Zweback's away. And I just wanted to say a couple of things, just in case you haven't noticed. We posted the solutions for P-set 11. And then also later in the week, we'll post the solutions for the extra problems that came along with P-set 11, so you can look at those.

And also, there's two past exams with solutions also on the website now. So you can start going through those. And also, there's a formula sheet there. And if you've got suggestions for things that you think should be on there that aren't, let us know and they probably can be put on there.

So I want to turn back to what we were doing at the end of last lecture, which was talking about the spin-orbit coupling. And so this is a contribution to our Hamiltonian that looks like spin of the electron dotted into the angular momentum that the electron has around the proton in the hydrogen atom. And so because of this term we had to change the complete set of commuting observables that we wanted to talk about.

So we have now this full Hamiltonian that includes this piece that has the Se dot L term in it. We have L squared, we have the spin squared. But because of this piece, Lz, which was previously one of the quantum numbers we used to classify things by, that doesn't commute with this term, right? So here we have to throw that one away.

Similarly, we have this throw away the z component of the electron spin. That doesn't commute with this either. And what we replace those by is actually the J squared and the Z component of J. So J is the vector sum of the angular momentum and the spin of the electron. And this is very interesting. This term does something interesting.

So if we look at-- let me go up here. If we remember the hydrogen states when we don't have this term, there's a state that has n equals 2 and l equals 1. And you can think of that as three states. And then we've got to tensor that with the spin, so the spin of the electron could be spin up or spin down. So there's a spin a half, so this is two states. And so you've got a total of six states you're going to talk about.

And now what we have to do is classify these according to the quantum numbers that are actually preserved by the system. So we can't use Lz or Sz. We have to use J squared and Jz. So we've got a J equals 3/2 multiplet-- and that's four states-- plus a J equals 1/2.

And you can see the number of states works out. We've got 3 times 2 is equal to 4 plus 2. And so this L dot S term takes these original six states, which without this interaction degenerate, and it splits them into the four states up here, and then two states down here. The J equals 1/2, J equals 3/2.

And we also worked out the splittings. If I do this, this is plus h bar squared over 2. And this is minus h bar squared. So this gives you a splitting.

Now this is not the only thing that happens in hydrogen, because you probably all know that the proton itself has spin. The proton has a spin 1/2 particle, just like the electron. It's even more complicated because it's a composite object. But that leads to additional splittings in hydrogen. And so these ones, this one here is called the defined structure. Or we can also talk about the type hyperfine structure.

So this is going to be a small effect on top of this one. So we have the proton that's spin 1/2, we have the electron spin a half, and then we have the relative orbital angular momentum. And so the total angular momentum, which is J, which is going to be the sum of L plus the spin of the electron plus the spin of the proton, this is conserved.

And the thing we were talking about here is actually not conserved. So once you worry about the spin of the proton you've got to look at the total angular momentum. And that's what will be conserved. And so our complete set of commuting observables is going to be a four Hamiltonian, which we'll get to in a moment, L squared the spin squareds of the proton and the electron, and then J squared, and finally Jz is the things that we're going to end up classifying states by.

So we originally thought about these two here, and did a coupling between those. It's pretty natural to assume that there maybe couplings between the angular momentum and the spin of the proton, which there are. But also there's going to be a coupling between the spins of the electron and the proton. And that's the one we're going to talk about at the moment. The other one is there but we won't go over it in any detail.

So the proton and the electron both spin 1/2 particles, and they both have magnetic dipole moments, which are proportional to their spin. And so it's really a coupling between these moments that tells us what the effect of this interaction is going to be. So we have the mu of the electron is equal to e over me-- minus me-- times the spin of the electron. And mu of the proton is, let me just write it as gp.

And gp happens to have the value of about 5.6. And this is actually kind of interesting. So if you look at the formula up here, really I could have written this as a g over 2, with g being 2. So for the electron, the g factor is very close to 2. This is because the electron is essentially a fundamental particle, with no substructure.

But the proton, which is made up of quarks and gluons flying around inside some region, has a lot of structure. And so this is really indicative of it being a composite particle. Because a fundamental spin 1/2 particle should have this g being 2.

So we've got these two dipole moments. And one way to think about this is you've got this dipole of the proton. We're going to think about the proton having a little dipole charge-- sorry, dipole magnetic moment-- and this produces a magnetic field. And the electron is sitting in that magnetic field. And its spin can couple to the field.

So we're going to have a Hamiltonian, a hyperfine Hamiltonian, that looks like minus mu of the electron dotted into a magnetic field produced by the proton, which is going to depend on r, on where the electron is. And you can simplify this as just e over m spin of the electron dotted into this B of the proton.

So we need to know what this dipole field is. And for that you really have to go back to electromagnetism. And you've probably seen this before. But let me just write it down, and we won't derive it here. But let's go down here.

This has a kind of complicated form. So there's this piece, and then there's another piece that looks like 8 pi over 3c squared mu p times the delta function at the origin. And so you think about the dipole field arising from a spinning charge distribution here. So we've got a magnetic dipole moment pointing up. This produces a field like this, a dipole type field going this way. So this is our B.

And then inside here, you should really think of taking the limit as this thing goes to 0 size. And so in order to get the right field in the middle, you need to have this term here.

And so if you want to see this being derived you can look in *Griffiths.* That does the derivation of this. But we will skip that.

So we've got the field. And now we can put it into our Hamiltonian. So it's mu e. So I could replace my mu's with the spins. So I get some factor out the front that looks like ge squared over 2 Me Mp c squared. And then I get 1 over r cubed plus--

So just plugging those in we get this Hamiltonian here. And let me just simplify a little bit. Let's just call this thing q.

And so this Hamiltonian is going to be given by q. And I can write it as the i-th component of the electron spin, the j-th component of the proton spin, dotted into r hat i hat j minus-- So just taking the common factors of the spins components out the front.

So if we've got this, we want to ask what it's going to do to the energy of the ground state of hydrogen. So we're going to take matrix elements of this between the hydrogen wave functions. So does anyone have questions so far? Yes.

AUDIENCE: Can you use r as a [INAUDIBLE]?

PROFESSOR: Right right. So these are unit vectors in the r direction. And this r is the length of the vector, r vector. The usual thing.

So what we're going to try and evaluate is the expectation value. So we're going to do this. Because going back to the start of last lecture, this is going to be a small correction. And so we can work out its contribution to the energy by using the original wave functions, but just calculating its matrix elements. So we're going to calculate-- and let me just give this a name. This can be--

So this is q, and the ground state has no angular dependents. So in fact, for the ground state, I can just write this is a function of r squared. For overtly excited states I can't do that. But for the ground state that works.

And then we have, so we've got the wave function. And then in between them we have to put this stuff over here. So let's put the there. So one of these terms is very easy to evaluate. With this [INAUDIBLE] function I just get the wave function at the origin. And the second term is actually also relatively easy to evaluate.

Who can tell me what this integral over all three directions of just one direction? What's that?

AUDIENCE: 0.

PROFESSOR: 0. And you can argue that by just asking, well what can it be? It's got to carry an index, because there's an index on this side of the equation. And there's no other vectors around in this problem. So the only thing it can be is 0.

So if I do integral d3r of ri rj, what can that be? Sorry?

AUDIENCE: 1.

PROFESSOR: 1? No. So it's got two indices. So the thing on this side of the equation also has to have two indices.

AUDIENCE: Delta ij?

PROFESSOR: Delta ij, very good. So the only thing that can carry two indices is delta ij. And then there might be some number here.

And it actually turns out that you can do an even more complicated integral. We can look at integral d3r of ri rj sum f of r squared. And that is also just some number, which depends on what f is, times delta ij.

And if you go along these lines and actually look at this, the difference between the integral of this piece and the integral of this piece is actually a factor of 1/3. And so this actually integrates to 0. So when I integrate over this one, I get something times delta ij. And that something is actually 1/3.

And so this term and this term cancel in the integral. And so you just get the delta function contributions. So you get some number times Sei S delta ij. So it becomes Se dotted into Sp. 8 pi over 3. And then it's psi 100 at the origin.

So this we know, we've already computed these radial wave functions, and saw at the origin this one is actually 1 over pi times the Bohr constant. And if you plug-in what Q is, and what the Bohr constant is, you can just find out that this whole thing ends up looking like 4/3 this gp and this we can call delta e hyperfine.

So you end up with a very simple thing. And it's just proportional to the dot product of the two spins. So you've seen, essentially, you saw this term in your homework. So we just assume that this thing here came out of nowhere and was just some number times Se dot Sp, and this was a contribution to your Hamiltonian. But now we actually know where that comes from.

And interestingly, this thing here, this whole thing, it's still an operator because it's got these spins in it. And that's-- put a star next to that because it's important.

So now we need to ask, well what are the real states of hydrogen so they're where we've got two spins? The spin of the proton, they could be aligned, or they could be anti-aligned. Oh, sorry. We have a question up there.

AUDIENCE: Is that np over np, or mu u?

PROFESSOR: No, me, mass. Mass of the electron over mass of the proton. So you have to remember that the spins of the proton and the electron to can parallel or they can be anti-parallel. Or they can be both down.

And so we have to go back and work out-- we have to realize that because of these terms the z components of those spins are not good quantum numbers. The only z component that appears in our list is Jz, so the total z component of angular momentum.

So we need to go back and do what you-- you probably have done this to the P-set. But let's just do it very quickly. We'll take those two spin 1/2 things and so let's make this J1 and this is J2. And we're going to have J.

So if I've got these two spins I can make various things. I can write down-- And if I've done this than I should also write that the m, the m quantum number that goes with the J quantum number is going to be equal to m1 plus m2. So this state here, because both of the spins are pointing up, this is an m equals 1 state. And then we can also have something like 1/2, 1/2.

You could have these two states. So they both have m equals 0. And then there's an m equals minus 1, which is 1/2 this guy.

So since this has m equals 1, and 1/2 cross 1/2 is going to give us a spin 0 multiplet and a spin 1 multiplet, because it's got m equals 1, this has to be J equals 1 as well. And this one has to be J equals 1.

But the two states in the middle, we don't know what those are. There's going to be a J equals 1, m equals 0 state, which is going to be some linear combination of these two. So let's just go over here. We don't need any of this. And we need to work out what the linear combination is.

So something to remember is this. The J plus or minus acting on Jm is this funny square root thing. So these are the raising and lowering operators. They take us from one state to the one with a different m value.

And so we can use that to start with. We could basically take J minus on our state. And according to this formula, this will give us the square root of 1 times 2 minus m is 1. And this should be 0, right?

I think I've got this sign up the wrong way. I think this is minus plus. No, sorry, that's right. It should be-- I'm doing the J minus so I have 1 times 1 minus-- yeah, right, so it's this.

So this is square root 2 times Jm minus 1. But we also know that J minus a is equal to J1 minus plus J2 minus because J is just the vector sum of the two J's. So we can ask what J minus on the state is. But this state we can write in terms of the tensor product. So this is equal to J1 minus 1.

If we use this formula for lowering something with spin 1/2 we get 1/2 times 3/2 minus 1/2 times minus 1/2, which is actually 1 under that square root. And so this actually equals 1/2 minus 1/2 tensor 1/2, 1/2. So these two things are equal.

And so that tells us, in fact, that the 1, 0 state, which is what's over here-- oh, sorry. Oh, why did I do that? This should be J equals 1 and M equals what it was, minus 1, this.

So the 1, 0 state, if we bring that 1 over root 2 on the other side is this combination. So it's one linear combination of those two pieces.

We also want the other one. So we've got three of our states. The fourth state is then, of course, the other linear combination of the two states over there. And so that's going to be our J equals 0, M equals 0 state.

So this state is going to be orthogonal to the one we've just written here. And so this is pretty easy to work out. Since there's only two terms, all we do is change the sign of one of them and it becomes orthogonal, because these states here are normalized. So this becomes 1/2, 1/2 tensor-- let me just write it in the same way that-- 1/2 minus 1/2 minus--

And so our four states, so we can condense our notation so we can say that this state we can just label as this. And we can just label as a down arrow. And then something like we can label as just up down, just to make everything compact. You just have to remember that this is referring to the first spin, this is always referring to the second spin.

And so then we can write our multiplets. J equals 1 has three states. It has up, up. It has up, down plus down, up. And it has down, down. So those are our three states that have J equals 1. And then we have J equals 0, which just has 1 over square root 2 up, down minus down, up.

And so the two spins in our hydrogen atom, the spin of the proton and the electron, can combine to be a J equals 1 or a J equals 0 system. And since we're talking about the ground state of hydrogen, it has 0 angular momentum. And so I'm really just talking about J total, here.

So if we now have this Hamiltonian, which is still an operator in spin-- we've dealt with the spacial dependence of the wave functions, but it's still an operator in spin-- we can now evaluate this. So we can take its expectation value in either the J equals 1 multiplet or the J equals 0 multiplet.

So let's just write it out again. So we have h hyperfine 1, 0, 0. This is equal to some delta E HF spin of the electron dotted into the spin of the proton.

We can rewrite this using something we did last time. We can write this as J squared minus Se squared minus Sp squared, with the 1/2 out the front. So here, because l equals 0 because we're in the ground state, then J equals Se plus Sp. And so J squared is going to give us Se squared, Sp squared, and then the dot product.

So great. So what is this, the spin squared of the electron? What's the eigenvalue of J squared, always? J, J plus 1 times h bar squared. And what is J for the electron? 1/2. And what about the proton?

AUDIENCE: 1/2 as well.

PROFESSOR: 1/2. So we've got 1/2 times 1/2 plus 1, so 3/2. So this gives us minus 3/4. This gives us minus 3/4. So this just looks like delta e HF over 2 J squared minus 3/2 h bar squared. OK? Anyone lost doing that? Or is that OK?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yep.

AUDIENCE: So, when you define delta e [INAUDIBLE] over there, that exudes energy?

PROFESSOR: Oh, you're right. You're very right. I've messed up. I've--

AUDIENCE: [INAUDIBLE]

PROFESSOR: Let me see. Yeah, really I have an h bar squared here. I think I should have had an h bar squared over there as well. Yeah. That should be over h bar squared here. Thank you. OK so--

AUDIENCE: [INAUDIBLE]

PROFESSOR: Sorry?

AUDIENCE: When does it get [INAUDIBLE]

PROFESSOR: That was just in the algebra going from this expression, writing it in terms of alpha, things like that. So it's just some algebra. OK, anything else? No? Good.

OK so now we can easily evaluate these things. We can now take J equals 1 and some M-- and this is for M equals all three states here-- and just evaluate this. And all that means is we have this J squared operator acting on this state here. And this gives us h bar squared 1 times 1 plus 1, or 2h bar squared.

So this will give us delta e hyperfine over 2. And then we've got, let's pull the-- sorry there's still and h bar squared here, and an h bar squared there. But now we can evaluate. The h bar squared here cancels that one, and we get a 1 times a 1 plus 1 minus 3/2. And that's just one quarter, which is--

And similarly we can take the J equals 0 state, and this one gives us delta e hyperfine over 2. And then it's 0 time 1 minus 3/2. And so that equals minus 3/4 EHF.

So what we're doing is evaluating these in these particular J states. And now we end up with something that's just a number. It's no longer an operator. It's an energy that we can measure. Yeah?

AUDIENCE: So, this expectation value h hyperfine 1, 0, 0, is still an operator. Is that because we only took the expectation value over the angular [INAUDIBLE]

PROFESSOR: We took over the spacial wave function. We did the r integral, right? But we didn't--

AUDIENCE: [INAUDIBLE]

PROFESSOR: Right, right. Yeah.

So this is actually a really important system. So let's just draw the energy level diagram here. And here we have four states. We have the spin 1/2 times spin 1/2. So 2 times 2 states. So we get a triplet and a singlet.

And what this hyperfine splitting does is take those four states and split the triplet up here, and split the singlet down here. And this gap we can see is-- oops, so this should be a delta HF. So this gap is delta E HF. So it's 1/4 and minus 3/4.

And if you plug numbers in, delta E HF, this actually ends up being 5.9 times 10 to the minus 6 electron volts, which is a pretty small scale. So you should be comparing that to the binding energy of the ground state of hydrogen of 13.6 electron volts. So this is a very small effect. And you can really think about the relative size. So the Bohr energy, so that 13.6, formally this goes like, alpha squared times Me c squared.

Then last time we talked about the S coupling, so the spin orbit, or fine structure. And so this one we found went like alpha to the fourth Me C squared. So smaller than the binding energy by a factor of 1 over 137 squared, or about 20,000.

And then this one that we're talking about here, the hyperfine, this, if you look over here, this is going like alpha to the fourth Me C squared times an additional factor of Me over Mp. And the mass of the proton is about 2,000 times the mass of the electron. And so this again is-- oh, sorry. This is alpha to the fourth. So this is suppressed by about another factor of 2,000.

You can go further. There are further corrections to this in something called the Lamb shift, which we won't say anything else about. This goes like alpha to the fifth Me C squared. And there's a whole host of higher order corrections. People actually calculate these energy levels to very high precision. But we won't do any more.

So this transition here is actually astrophysically extremely important. So if we think about something sitting in the state here, it can decay down to the ground state by emitting a photon. So we can decay from J equals 1 to J equals 0 by a photon.

And that photon will have a wavelength that corresponds exactly to this energy difference. And so that wavelength is going to be, we can write it as c over the frequency, or hc-- oh, hc not h bar c-- hc over this delta e hyperfine. If you plug numbers into this you find out that this is approximately 21.1 centimeters. And the frequency is 1,420 megahertz. And so right in the middle-- well, at the end-- of the FM band in radio. So theses are radio waves.

So the size of this wavelength is firstly important because it's large compared to the size of dust in the universe. So dust is little stuff. So this is essentially goes straight through dust. So these photons will go straight through dust.

The other important thing is that you probably know that there's a cosmic microwave background radiation in the universe, that's essentially very close to constant everywhere. And so we have, essentially, we have a temperature of 2.7 Kelvin. That corresponds to photons with an energy kT, which is about 0.2 times 10 to the minus 3 electron volts. So milli electron volts.

But if you compare this number to what we have here, this cosmic background microwave radiation can excite hydrogen from here up to here. There's enough energy for one of those photons to come along, knock the hydrogen atom, and excite it up to here. And then it will decay and will emit this beautiful 21 centimeter line, which will go through all the dust. And so we can actually see the universe in this 21 centimeter line.

Even more remarkable is, we can't calculate this at the moment, but you can show that the lifetime for this transition to happen is about 10 to the 7 years. So we can never measure that in a lab. But because these hydrogen atoms can be wandering around the universe, not interacting for that long, then they can emit. And so we can see that.

This was first observed in about 1951, and is the first way that we actually saw that the galaxy had spiral shaped arms. So it's pretty important. And another nice thing about this is if you think about another galaxy-- so let me just draw another galaxy, a spiral galaxy somewhere else, like this.

Let's have us over here looking at this galaxy from side on. This galaxy is rotating. So this one's moving this way, this one to moving this way. There's hydrogen over here and over here. And so we get these photons coming over here, and photons coming to us from there.

But what's going to be different about these?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Their rate shifted, right? The Doppler shifted. So this is my 21 centimeter photon. But they get Doppler shifted. And so we can measure the difference in the frequencies of those. What does that tell us?

AUDIENCE: [INAUDIBLE]

PROFESSOR: How fast this galaxy is spinning, right? And so one very interesting thing you find from that is if you look at the galaxy and count how many stars are in it, and essentially work out how massive that galaxy is, the speed of rotation here is actually-- that you measure from these hydrogen lines-- is that it's actually faster than the escape velocity of the matter. And so if all that was there was the visible matter, then the thing would just fall apart.

And so this actually tells you that there's dark matter that doesn't interact with visible light, that's kind of all over here. So that's kind of a pretty interesting thing.

So that, I think, yeah. So any questions about that? We're going to move on to another topic. Yep?

AUDIENCE: You said earlier [INAUDIBLE] it's 10 to the 7 years. Does that mean it takes an average of 10 to the 7 years for the cosmic microwave background energy to shift back [INAUDIBLE]

PROFESSOR: No, it's really, if I just took hydrogen in this state, and took a sample of it, that's how long it would take for half of it to have gone and made the decay. So it can happen much faster.

And there's a lot of it in the universe. So there's many more than 10 to the 7 atoms of hydrogen in the universe. So we see more than one of these things every year. So if you were just looking at one of them you would have to wait a long time.

AUDIENCE: But the thing about the cosmic microwave background is to go from [INAUDIBLE] from 0 to 0 up to get [INAUDIBLE]

PROFESSOR: Right so I mean the energy is large compared to that. So it will typically knock you up into an even higher state. And then you will kind of decay down. But then this last decay is-- because this lifetime is very long, the width of this line is also very, very narrow.

So now let's talk more about adding angular momenta. Oh, maybe I should have left that up. Too late.

So we're going to do this in a more general sense. So we're going to take J1, some spin J1 that has states J1, M1, with M1 equals minus J up to J. And so we're sort of talking about something like the electron in the hydrogen atom. And so that's not in any particular orbital angular momentum.

So we can talk about that the Hilbert space that this thing lives in. So we can think about particle 1 with angular momentum J1. And this is basically spanned by the states J1, M1 of these [INAUDIBLE].

We can take another system with another J2, and this is going to have states J2, M2, with M2-- that should be J1's there. And that would similarly talk about some Hilbert space of some fixed angular momentum.

If we want to talk about the electron in a hydrogen atom, where it doesn't have a fixed angular momentum, what we really want to talk about is the Hilbert space H1, which is the sum over J1 of these Hilbert spaces. And so this is talking about-- this Hilbert space contains every state the electron can have in a hydrogen atom. It can have all the different angular momenta.

And similarly we could do that for J2. We can define J, which is J1 plus J2, as you might guess. And really this you should think of as J1 tensor the identity plus the identity tensor J2, where this one is acting on things in this Hilbert space, and the 1 here is acting on things in this Hilbert space. And similarly there's an H2, J2 that goes along with these guys.

And so this operator, this big J, is something that acts on vectors in things in this tensor product space. Actually I should label this with a J1. It also acts on things in the full space, but we can talk just about that one.

So now we might want to construct a basis for this space. And we conversely construct an uncoupled basis which is just take the basis elements of each of the spaces and multiply them. So we would have J1, J2, M1, M2. We'd have the states here. And if we just ask what our various--

J1 just gives us h bar squared J, J plus 1. And this one gives us h bar M1 h bar squared times our state. And so we can think about all of these. And this is what we label our state with. And that's because these form a complete set of commuting observables. And we'll just call this the A set.

We can also talk about our operator J and use that to define our basis. And let's just be a little explicit about what J squared is going to be. So this is J1 tensor identity plus 1 tensor J2. And the same thing here. If you expand this out you get J1 squared tensor identity plus 1 tensor J2 squared plus the dot product, which we can write as the sum of J1k tensor J2k.

And because of this piece here, J squared doesn't commute with J1z, for example. So we can't add this operator to our list of operators over there. And similarly J2z J squared is not equal to 0. So if we want to talk about this operator we have to throw both of those away.

But there is an operator total Jz that commutes with J squared. And it also commutes with J1 squared and J2 squared. And so we can have another complete set of commuting observables B that's equal to J1 squared, J2 squared, J squared, and Jz.

And so if there are observables then the natural basis is to label them by the eigenvalues here. So we're going to have a J1, a J2, then a J and an M. And so this is the coupled basis.

Now both of these bases are equally good. They both span the full space. They're both orthogonal, orthonormal.

And so we can actually write one basis into in terms of the other one. And that's the generic problem that we are trying to do when we're trying to write what we did over here before, when we did spin 1/2 cross spin 1/2. We're trying to write those products states in terms of the coupled basis.

Well, they're both orthonormal basis. So I can expand J1, J2. Well actually, maybe I'll say one more thing first. So being orthonormal means that, for example, sum over J1, J2--

This is 1, right? You can resolve the identity in terms of these states. And this is the identity on this Hilbert space.

I can also think about the identity just on this smaller Hilbert space, where the J1 and J2 are fixed. And so I can actually write it's the identity operator. So because every state in this space has J1 equal to some fixed value and J2 equal to some fixed value, then an identity in that thing is just somewhere over the M's, because they're the only things there.

So using this, because I know that the state J1, J2, Jm has some fixed value of J1 and J2, I can write this as a sum. I can use this form of the identity.

So I've written my coupled basis in terms of the uncoupled basis here. And these are just coefficients. These are called Clebsch-Gordan coefficients. They're just numbers like square root 2 and things like this. And I tell you how to do this decomposition.

So they have various properties. Firstly, sometimes you also see them written as C of J1J2J colon M1M2M and various other notations. So basically things with six indices are probably going to be these guys.

So they have various properties. The first property is they vanish if M is not equal to M1 plus M2. And this is actually very easy to prove.

So remember that Jz is just going to be J1z plus J2z. So as an operator I can write Jz minus J1z minus J2z. And what is that operator? It's just 0, right? This is equal to that. So this is 0. So I can put this 0 anywhere I want and I'll still get 0.

So let's put this between-- so this is 0-- put it between J1, J2, Jm on this side. So a coupled state here. And on this side I'll put it between the uncoupled state, J1, J2, M1, M2.

So this state is an eigenstate of Jz. And this state is an eigenstate of J1z and J2z. So I can act to the right with Jz, and act to the left with these J1z and J2z, and they have mission operators. And I know because this is 0, this whole thing is 0.

So then act this one on these guys and these two back this way. And so you see that gives me h bar. And then I get this one acting on here gives me M. And J1 acting on here gives me M1. And if M is not equal to M1 plus M2, then this term isn't 0. But the whole thing is so that has to be 0. So that's QED.

The second property is that-- So they only allow values of J fall in this range here. And each J occurs once.

Now one way to think about this is to think of these things as vectors. So you have vector J1, and then from the point of this you can have vector J2. But it can go into an arbitrary direction. So it can go up here, or it can go like this. These are meant to be the same length. And I can come all the way down here.

But I can could only sit on integer points. And so this is kind of J2. And so the length of this thing here would be the length of J1 plus the length of J2. So it would be this. And then the length of up to here would be this one. And then all of the other ones are in between.

But you can also just look at the multiplicities of the different states. So if we look at the uncoupled basis-- so the first state, which was J equals J1, there are two J1 plus 1 states, because it can have all of the M values from J1 down to minus J1. And the other one can have two J2 plus 1 states. So that's the total number of states that I expect to have.

So now let's assume this is correct and ask what the N coupled is. So this would be the sum over J equals mod J1 minus J2 up to J1 plus J2 of 2J plus 1. And let's assume that J1 is greater than or equal to J2, just to stop writing absolute values all the time.

So we can write this as the difference of two sums, J equals 0 to J1 of-- J1 plus J2-- of 2J plus 1 minus the sum of J equals 0 to J equals J1 minus J2 minus 1 of 2J plus 1. And if you go through-- so this is just N,N plus 1 over 2 for each of these things. You end up with, well, you end up with this. You end up with the same thing. And so this is at least consistent that the number of states that we have is consistent with choosing this.

One other thing we can do is look at the top state, and just see if that works. See if that has the right properties. So because we know that the J1, J2, J equals J1 plus J2, M equals J1 plus J2. So the maximal state, the only way we can make this is to take J1, J2, M1 equals J1, M2 equals J2.

Our spins are completely aligned in the total up direction. Yeah?

AUDIENCE: Sir, would you be able to write a little larger?

PROFESSOR: Yes, sorry. OK, yeah. That's why I like a big chalk. But we've run out of big chalk, so I'll try.

So we know, also, that J squared is equal to J1 squared plus J2 squared plus the dot product. We can write that out as J1 squared plus J2 squared plus 2J1z J2z plus J1 plus J2 minus plus J1 minus J2 plus.

And then we can ask what does J squared on this state give? And this is J1, J2, J1 plus J2, J1 plus J2. So J squared, so we know what that should be. That should return J1 plus J2 times J1 plus J2 plus 1 times h bar squared, because J is the good quantum number. But let's let it act on this piece.

So this equals J1 squared plus J2 squared plus 2J1z J2z plus J1 plus J2 minus plus J1 minus J2 plus acting on J1, J2, J1, J2. So this state here.

So we know how that acts. So this one gives us-- everything gives us an h bar squared. This gives us J1 J1 plus 1, for this term, plus J2 J2 plus 1 for the second term. Each of these gives us the M quantum numbers. But that's J1 J2. So this is plus 2J1, J2. And now what does this one do? J1 plus on this state.

AUDIENCE: Kills it.

PROFESSOR: Kills it, right. Because it's trying to raise the M component of 1, and it's already maximal. And this one, J2 plus, also kills it. So you get plus 0 plus 0 times the state.

So if you rearrange all of this you actually find you can write this as J1 plus J2, J1 plus J2 plus 1 times the state, which is what you want. So the J squared operator acting in the coupled basis gives-- well, acting in the uncoupled basis gives you what you expect in the coupled basis.

So now I need a big blackboard. So let's do an example of multiplying two things. So let's write out a multiplet. So we're going to take J1, and we're going to have J1 bigger than J2 here. So we've got J1 J1, J1 J1 minus 1. And then somewhere down here I've got J1 and 2J2 minus J1. And then all the way down to J1 minus J1.

So this has two J1 plus 1 states. And I'm going to tensor that was another multiplet, with my J2 multiplet, which is going to be smaller. So I'm going to have J2 J2. Oh, maybe I'll put one more in. And down here we've got J2 comma minus J2.

And so here we have two J2 plus 1 states. And importantly, this left hand side has two J1 minus J2 more states than the right hand side. Just counting those states that's pretty obvious.

So now let's start multiplying these things, and forming states of particular values of M, the total M. So if we say we want M equals J1 plus J2 what can we do? How can we make that?

So we have to take the top state in each case. Because if I take this one and I take this M value, I can't get up to this, right? So there's only one way to make this. So I'm going to draw a diagram of this. We're going to have a one state there.

The next M value, J1 plus J2 minus 1, how can I make that? So I can start with this state, and I will multiply it by this one, right? Or, what else can I do?

AUDIENCE: Start with the second down on the left and tensor with the top?

PROFESSOR: That's right. So I take those two. So there are two states. And those two states are just two linear combinations. So let me draw two dots here, I can form two states.

Keep going-- minus 2, I get three states. And let me try and draw lines here to guide this stuff.

OK, I'm not going to keep going. But at some point we get-- what's the largest number of states of a given M I can make going to be? Can anyone see that?

AUDIENCE: 2J2 plus 1?

PROFESSOR: 2J2 plus 2, right, because I've got 2J2 plus 1 states here. And I'm taking one of these plus one of these will give me-- so down here I'll have an M equals-- what is it-- J1 minus J2. And here I have 2J2 plus 1 states.

And so let me kind of draw some of these states in. And then dot, dot, dot. And then over here we end up with this guy. So if I go down to the next one, how many states?

So to form those states I was taking this top state with the bottom state here. That gives me J1 minus J2, right? Or I was taking the second state here with the second to bottom state here, and so forth. And then all the way up to here.

Now if I then start shifting things down in this side, but leave exactly the same things over there, then I'll lower J by 1. And I'll keep doing it until I hit the bottom. And because there's this number of states more in the right hand side than the left hand side-- hang on, let me just write this one.

OK, we might need to go onto the next board. So this keeps on going until I get to M equals-- I don't remember the number-- M equals J2 minus J1. And there are 2J plus 2 plus 1 states of this.

And then once I do that, then I start having fewer and fewer states. Because I've gone basically moving out the bottom of this multiplet. And so here we have, this is 2 J1 minus J2 plus 1 rows. And then I start contracting.

So the next one, M equals J2 minus J1 minus 1 has two J2 states. So then we can keep going. And this is meant to continue this diagram up here. So then we keep going down, down, down. And then we'd have M equals minus J1 plus 1. And how many states can I make that have that?

So I need to take this one, and I could take the-- oh, where is it? Sorry, not this. This is not what I mean. Minus J1 minus J2 plus 1. That's more obvious, right? So there's two states.

And so this picture has kind of-- this line starts coming in. And now I've got my two states here. I have the next one, I've got three states. And then finally, M equals minus J1 minus J2, I have one state. And so I get this. And so you actually-- oh, that was not very well drawn.

So if you look how many states there are in this first column, how many is going to be there? So it goes from plus J1 plus J2 to minus J1 minus J2. So there's two J1 plus J2 plus 1 states there. And here there is two J1 plus J2 minus 1 plus 1 states in this guy.

And so this is a J equals J1 plus J2. This one is J equals J1 plus J2 minus 1. And if you are careful you'd find that this one here, the last one here, this has this many states in it, two J minus 1 plus 1 states. So this is a J equals J1 minus J2. And to be completely correct, we put an absolute value in case J2 is bigger than J1.

So this is our full multiplet structure of this system. So all of the states in this column will transform into each other under rotations, and things like this. And same for each column, they all form separate multiplets.

So just some last things before we finish. So another property of Clebsch-Gordan coefficients we can choose them to be real. They satisfy a recursion relation but don't have a nice, closed form.

I think this is in *Griffiths*. It gives you what this recursion relation is. I think it does, at least many books do. And also, they're tabulated in lots of places. So if you need to know the values, you can just go and look them up, rather than trying to calculate them all necessarily.

And I think that's all we've got time for. So are there any questions about that? Any questions about anything else? OK, great. So we will see you on Wednesday for the last lecture.