Quantum Physics III (8.06) — Spring 2016

Assignment 6

1. **Born Approximation for Scattering From Yukawa and Coulomb Potentials, plus a Practical Example of the Latter (25 points)**

   Make sure you are aware of Griffiths' Examples 11.5 and 11.6 on page 415 as you do this problem. He has done some of the work for you.

   Consider a Yukawa potential

   $$V(r) = \beta \frac{e^{-\mu r}}{r}$$

   where β and μ are constants.

   (a) Evaluate the scattering amplitude, the differential cross section $d\sigma/d\Omega$, and the total cross section in the first Born approximation. Express your answer for the total cross section as a function of the energy E.

   (b) Take $\beta = Q_1 Q_2$ and $\mu = 0$, and show that the differential cross section you obtain for scattering off a Coulomb potential is the same as the classical Rutherford result. Use this differential cross section in part (d) below.

   (c) Differential cross sections are what physicists actually use to calculate the rate at which scattered particles will enter their detectors. The number of particles scattered into solid angle $d\Omega$ per second by a single scatterer is given by

   $$\frac{d^2N}{dt d\Omega} = \frac{d\sigma}{d\Omega} \times \frac{d^2N}{dt dA}$$

   where $d^2N/dtdA$ is the incident flux in units of particles per second per unit area, ie per unit cross sectional area transverse to the beam. Consider a uniform beam of dN/dt particles per second with a cross sectional area A. This beam strikes a target with density n (n is the number of scattering sites per unit volume) and thickness t.

   Give an expression for the number of particles scattered into a detector with angular size $d\Omega$ per unit time.

   Show that your result is independent of the cross sectional area of the beam even if the beam is not uniform across this area. [Note that this is important, because it is typically easy for an experimenter to measure dN/dt but hard for her to measure either A or the uniformity of the beam across the cross sectional area.]

   (d) Consider a beam of alpha particles ($Q_1 = 2e$) with kinetic energy 8 MeV scattering from a gold foil. Suppose that the beam corresponds to a current of 1 nA. [It is conventional to use MKS units for beam currents. 1 nA is 10^{-9} Amperes, meaning 10^{-9} Coulombs of charge per second. Each alpha particle has charge $2e$, where $e = 1.6 \times 10^{-19}$ Coulombs.] Suppose the gold foil is 1 micron thick. You may assume the alpha particles scatter only off nuclei, not off electrons. You may also assume that each alpha particle scatters only once. You will need to look up the density of gold and the nuclear charge of gold (Q_2). How many alpha particles per second do you expect to be scattered into a detector which occupies a cone of angular extent ($d\theta = 10^{-2}$ radians, $d\phi = 10^{-2}$ radians) centered at $\theta = \pi/2$?

2. **Scattering from a spherical well (30 points)** For some parameters γ, b, consider the following spherically symmetrical potential:

$$V(\vec{r}) = V(r) = \begin{cases} -\frac{\hbar^2}{2m}\gamma^2 & r \le b \\ 0 & r > b \end{cases} \tag{1}$$

We will consider s-wave (i.e. $\ell = 0$) scattering off this by an incoming plane wave with momentum $\hbar k$.

(a) The radial Schrödinger equation is

$$\frac{d^2u(r)}{dr^2} + k^2u(r) = \frac{2m}{\hbar^2}V(r)u(r).$$

For $r > b$ this has solution $u(r) = \sin(kr + \delta_0)$. Write down a valid solution for $r \le b$. Use the $r = 0$ boundary condition to ensure that this solution has only one free parameter.

(b) Match boundary conditions and solve for δ_0 as a function of k and b.

(c) Compute the scattering length $a \equiv -\lim_{k\to 0} \frac{\tan(\delta_0)}{k}$. Plot a/b as a function of γb. (These axes labels are chosen so that both are dimensionless.)

(d) Your plot should have many zeros. For these values of γb we have $\sigma_0 = 0$ and there is no s-wave scattering. This is known as the Ramsauer-Townsend effect. Numerically find the smallest positive value of γb for which $a = 0$.

Your plot should also have infinities when $\gamma b = (n + 1/2)\pi$ for n a nonnegative integer. What happens to δ_0 and σ_0 at these points? Is this consistent with the bound from partial-wave unitarity?

(e) Let's try to explore these infinities more. In the above we took the $E \to 0$ limit from above, i.e. considering E to be positive and very small. Now consider $\lim_{E\to 0^-}$; i.e. suppose $E < 0$ and take the limit as E approaches zero. Now solutions to the Schrödinger equation correspond to bound states. We can equivalently think of k as $i\kappa$ for some $\kappa > 0$. Such a bound state with E very close to zero is called a "threshold bound state" because it is near the threshold energy for valid bound states. Which values of γb correspond to threshold bound states? For each such value of γb how many bound states (i.e. not only including threshold bound states) does the potential support?

For partial-wave scattering at fixed ℓ, the S-matrix (relating outgoing to incoming waves) is 1-by-1. When the S-matrix equals ∞ we can interpret it as a solution in which there is an outgoing wave but no incoming wave; this is precisely what happens for bound states if we allow the imaginary wavevector $k = i\kappa$.

(f) Sketch $u(r)$ as a function of r/b for $k = 0$ and γb equal to 0, $\pi/4$, $\pi/2$,π.

(g) Suppose γb is slightly larger than $\pi/2$, so there is a threshold bound state with energy $-E_B$. Show that for incoming waves of (positive but small) energy E, $\sigma_0 \approx \frac{c}{E+E_B}$ for some c. Find c.
This is an example of the general and widely useful principle that low-energy scattering can be used to detect bound states.

3. **Scattering from a δ-Shell (20 points)**

Consider s-wave ($\ell = 0$) scattering from the potential

$$V(r) = \lambda \frac{\hbar^2}{2mR} \delta(r - R)$$

with λ a large positive constant. To find the phase shift $\delta_0(k)$ we have to solve

$$\frac{d^2u}{dr^2} + k^2 u = \frac{\lambda}{R} \delta(r - R) u \ ,$$

with $u = 0$ at $r = 0$ and $u = \sin(kr + \delta_0)$ for $r > R$.

(a) What is u in $r < R$?

(b) By comparing $u'(r)/u(r)$ just inside and just outside $r = R$, find a formula to determine δ_0.

(c) Find the scattering length $a \equiv -\lim_{k\to 0} \delta_0/k$.

(d) Assume $\lambda \gg 1$. Sketch $\delta_0(k)$. Show that for kR just below $n\pi$, with n a positive integer, $\delta_0(k)$ increases very rapidly by π (as kR increases towards $n\pi$). Sketch the s-wave cross-section σ_0. Show that the s-wave scattering from this potential is the same as that from a hard sphere of radius R for all values of kR *except* those such that kR is close to $n\pi$. What is the significance of these values?

4. **Partial Waves (25 points)**

Suppose the scattering amplitude for a certain reaction is given by

$$f(\theta) = \frac{1}{k}\left(\frac{\Gamma k}{k_0 - k - ik\Gamma} + 3e^{2i\beta k^3}\sin 2\beta k^3 \cos\theta\right) \tag{2}$$

where Γ, k_0, and β are constants characteristic of the potential which produces the scattering. Of course $k = \sqrt{2mE/\hbar^2}$ is the deBroglie wavenumber.

(a) What partial waves are active (*i.e.* what values of ℓ)?

(b) What are the phase shifts in the active partial waves? Do they have the proper behavior as $k \to 0$?

(c) What is the differential cross section, $d\sigma/d\Omega$ for general values of k?

(d) What are the partial wave cross sections, σ_ℓ?

(e) Assume $\beta k_0^3 \ll 1$. Give an approximation to the total cross section $\sigma(k)$ for $k \approx k_0$.

(f) What is the total cross section for general values of k? What is the imaginary part of the forward scattering amplitude? Do they satisfy the optical theorem?

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