We're going to do a couple of results with this formula that connect even a little more with what we're doing. Maybe I'll leave that blackboard here, because [INAUDIBLE] simple formulas come out of this, which is very nice, actually. So one simple formula was that the total cross section-- we wrote the formula for the differential cross-section. But the total cross-section is the integral of fk of theta, in this case the omega. So this is the integral of fk of theta star fk of theta with omega.

But we have fk of theta here. That 0 looks like theta, isn't it? That's not good. That's a 0. And we can plug all this in. So what do we have? We do this because the answer is very simple. We get a 4 pi over k squared. We have a sum over l and l prime because we have two factors-- 2l plus 1, 2l prime plus 1.

The phase shift for l prime and the phase shift for l-- so e to the minus i delta l sine of delta l. So I guess I'm using l for this one and l prime for that one. e to the i delta l prime sine of delta l prime. But happily, all that will not matter, because then you have the integral over solid angle of yl star 0 of omega yl 0 of omega prime here.

Seems like a lot of work, but we're doing nothing else than integrating this f of theta. Let's look at it for a second. It's just that formula squared. So basically, you star the one y. You don't start the other, but you have to integrate them. And there orthonormality of our spherical harmonics means that this is delta ll prime.

So you can set l prime equal to l. These phases will cancel. This factor will be squared. This square root will disappear. And you have a simple formula that the cross-section is 4 pi over k squared sum 2l plus 1 sine squared delta l. Very nice formula for a physical observable.

What did you get? That the cross-section is the sum of contributions. The total cross-section is the sum of contributions from partial cross-sections from each partial wave. That's not true for the differential cross-section. Because in the differential cross-section, you don't integrate. And as long as you don't integrate, you have mixing between different l's. The different partial waves interfere in the differential cross-section, but they will not interfere anymore in the total cross-section. Each partial wave contributes to the total cross-section.

That's why it's important to calculate this phase shift. So you say, oh, just a phase. Phases don't matter. No. The relative phases matter a lot. Those are the phases that enter here, and the cross-section is expressed in terms of those phases.

There's one more result that is famous in scattering and has to do with the optical theorem. The optical theorem is something you may have seen already in electromagnetism or in other fields. It's a statement about probability of conservation of flux. And it's fairly non-trivial, and it's a constraint on the scattering amplitude.

Basically-- and we may discuss it in a problem. It's the statement that when you have an object, the thing that you detect as the scattering cross-section is all these particles that were deflected from the object. And they were deflected from the object because the object creates a shadow. At least in the electromagnetism, that intuition is very clear. You have a sphere here, maybe a conducting sphere, an observancy. You shine light. You create the shadow. And that is the light that if you didn't have here, it would have gone through.

But if it's here, the shadow is responsible. What you lost from the shadow is what you got scattered. So whatever you get-- in the forward direction here, you get nothing-- you get a shadow-- carries the information about the wave that scattered.

It's a little more complicated than that mathematically when you do it, because the total wave function in the forward direction is a combination of the incoming wave function that has some forward direction and the scattered wave function that has a forward direction. So the theorem is quite interesting to prove, and we prove it with flux conservation that gives you enormous insight into the physics.

But here is the power of algebra and the power of phase shifts. You don't have to be brilliant to discover the optical theorem in this setup. The physics has already done-- the mathematics of the phase shift has already done all the work for you. Let's see it happen.

Let's figure out how does the forward-- the scattering cross-section look in the forward direction. So you have your object here. You send in your waves. They get scattered. But there is something in the forward direction. That is fk at theta equals 0 is the forward scattering, the word "scattering."

And now this forward scattering is-- we can calculate it. For that, we need to know that yl 0 of theta equals 0-- well, yl 0 of theta is actually 2l plus 1 over 4 pi pl, the Legendre polynomial of cosine theta. So yl 0 at theta equals 0 is just square root of 2l plus 1 over 4 pi. And the Legendre polynomials are defined always so that pl at 1 is equal to 1. All pl's at x equals 1 are always 1.

So that's the spherical harmonic in the forward direction. So you have here from that formula on the right 4 pi over k sum from l equals 0. This is an investigation of what happens to the forward scattering amplitude. The yl 0 gives you another square root of 2l plus 1 and a square root of 4 pi as well. And then you get e to the i delta l sine delta.

So look here, here, and these factors simplify. So fk of theta equals 0 is 1 over k sum of l 2l plus 1 e to the i delta sine delta. That's a nice formula. Maybe something-- it has to do with a cross-section. f in this forward direction has that formula, and the cross-section has this formula.

So how do I get to relate them? I get to relate them if I set now-- I ask for the imaginary part of this fk. Because the imaginary part of this fk at theta equals 0 is equal to 1 over k sum over l, this 2l plus 1. And the imaginary part of that turns it into sine squared delta, because the imaginary part of the e to i delta is sine delta.

So I get the sine squared delta there. And you say, look at this sum. It's identical to the sum of the cross-section. You have discovered a relation between the total cross-section and the imaginary part of the forward scattering amplitude. So what is the final formula? The final formula is that sigma-- maybe I can do it here.

Sigma is equal to-- well, it differs at least from a 4 pi over k imaginary part of f of theta equals 0, fk of theta. This is the optical theorem, which was discovered in the context of optics by thinking of the physics of shadows. Whatever-- basically, the shadow contains all that kind of information about all that was lost from the incoming beam.

So that's a nice-- it's not absolutely obvious how it works, the process of interference between the incoming wave and the scattered wave and what they do. And you have to look carefully at this thing. So that explains why you need the imaginary part and why these factors show up. But the intuition is nevertheless simple.