MITOCW | L20.5 Identification of phase shifts. Example: hard sphere

PROFESSOR: I want to demonstrate two ways in which you can see phase shift. So basically, the reason we use phase shift is that these are the things that you can calculate. Calculating phase shifts is possible. So how do you do that?

We'll be a two-step procedure. We'll only finish that next time. But let's get started. So suppose you have your wave again. And for a fixed l, a given partial wave, this is the full solution for this scattering problem. It is an Al Jl of kr plus bl at Nl of kr. y l0. That's your solution.

The signal that you got scattering, and you have something right here, is the existence of this term, because when there's no potential and no scattering, the solution is valid all the way to r equals 0, and therefore, this has no singularity. But this term is saying that this solution doesn't extend all the way to r equals 0, because this diverges.

So something un-trivial is happening. So bl is the signal that they're scattering. Now, expand for large R. So this is proportional to Al sine kr minus l pi over 2 minus bl cosine kr minus l pi over 2 y l0 1 over a kr. It's the same. I'm going to drop all constants very fast.

Now, here is my claim. You were thinking of phase shifts. Well, the phase shift is nothing else than bl over Al is minus the tangent of the phase shift. This is a claim. Or you could say this is another definition of a phase shift, and I'm going to argue that it's the same, actually, than what we did.

To do that, I have to just expand a little more. So what I'm going to do is divide by a, so take the a out, akr. Now you have sine kr minus l pi over 2 minus bl over Al. Bl over Al is tangent delta, so plus tan delta l cosine kr minus l pi over 2 y l0.

And that is proportional to a over kr tangent delta l is sine over cosine. So let's put the 1 over cosine delta l here, sine kr minus l pi over 2 cosine delta l plus cosine kr minus l pi over 2 sine delta l y l0 of theta.

So that ratio tangent, I put a sine over cosine, I have it here. But this, your favorite trigonometric identity, is equal to a over kr 1 over cosine delta l sine, a single sine of kr minus l pi over 2 plus delta l.

So if this is the phase shift, the solution looks like this far away, a sine of kr minus l pi over 2, plus a delta l. That's one way of identifying the phase shift. But I want to show that's the same phase shift we had before, but that's clear already. Up to constants, this is-- I'm sorry, y l0 of theta. And up to constants, this is e to the ik r minus l pi over 2 plus i delta minus e to the i minus ik r minus l pi over 2 minus i delta.

And now I drop everything else. And now I multiply or take out this phase, take it out. I'm just working up to proportionality, which is all you care at this moment. And this is e to the i kr minus l pi over 2.

If I take that out, this becomes plus 2i delta l. And this becomes minus e to the minus i kr minus l pi over 2. And those are the waves we had before, somewhere here, here. Here they are. You see them? Still, they were here. This wave and that wave with delta here has showed up.

So this delta that I've defined here is the same phase shift we introduced before. So now you have three ways of recognizing a phase shift. A phase shift can be recognized in the partial wave expansion. A phase shift can be recognized by looking at the scattering wave far away and seeing that it takes this form, and you say, oh, here is the phase shift.

And the phase shift can be recognized by looking at the solution in terms of spherical Bessel functions, and it's the ratio of these coefficients. Those are the three definitions of the phase shift, three ways of seeing your phase shift. Instead of elaborating more on this, let's do one example to convince you that this is solvable and doable in fact.

So the example is a hard sphere example. This is the object that you're scattering off. The potential is equal to infinity for r less than a and 0 for r greater than a. This is the origin and radius a sphere, the waves come in.

You want the cross-section. OK. It might look like this is hard, OK. How are we ever going to solve this? In fact, will be very easy. We have everything ready to solve. So let's remember what we have. Well, there's going to be a radial solution, Rl Remember Rl is Ul over R. And that takes the form Al Jl of kr plus Bl Nl of kr.

That is the general solution for a radial thing, and therefore your general solution for your wave function or of theta is what we were writing here, except that the superposition of them. So I said here is how you recognize what is the phase shift for a given partial wave, but we're going to have all the partial wave. So this is going to be the sum over l of these things, Al of Jl of kr plus Bl Nl of kr times Pl of theta.

I could write y l0, but that's up to a constant Pl. So that's your general solution. And in fact, I didn't even have to write this, because you have there on that blackboard, that's the general solution for the full wave away from this sphere that's not valid for R less than 0. But this is your full solution for a given partial wave for all the waves it's there.

OK, so that equation solves a problem. pl of cosine theta, yes. It's really better written like that. Yes, that's more rigorous. Yes. OK, so what do we do now? Somehow you have to use it. You have a sphere. We haven't used a sphere yet. So what does the sphere tell you? It tells you that the wave function must vanish on the sphere, because it's infinitely hard.

It's like an infinite wall. So this wave function should vanish psi at a theta, which is equal to sum over l Al Jl of ka plus Bl of Nl of ka times Pl of cosine theta should be 0.

OK. One equation for infinitely many unknowns, but the Pl's are a complete set. If you expand the function of theta in terms of Pl's, you can determine every coefficient. They're linearly independent, the Pl's. So if you can think of this, this is an a. This is just a number. So this is a sum of numbers times Pl's must be 0. All the numbers must be 0, because the Pl's are independent.

Therefore, here we have that a Al Jl of ka plus Bl Nl of ka must be 0 for all l. And therefore, tangent delta l, which is minus Bl over Al has been determined. The tangent delta l is Bl over Al, and that's equal to-- it's Jl of ka over Nl of ka.

Done. All phase shifts computed. The Bessel functions are known. You look them up. You calculate them to any accuracy. But you have here all the phase shifts. Therefore, you have the cross section of this sphere. You have the differential cross section. You have anything you want. It has all been determined. We can do a little bit of algebra if you want to calculate what the sine squared delta of something.

Sine squared can be expressed in terms of tan squared. It's tan squared delta l over 1 plus tan squared delta l. So it's our Jl squared over Jl squared plus Nl squared. Tan l, you substitute this, and you get that. Therefore, the cross section can be calculated, and the differential cross section can be calculated. It's pretty interesting.

Let's do it here. The cross section, remember sigma is 4 pi over k, sum over l equals 0 to infinity 2l plus 1 times this sine squared. So Jl of ka squared over Jl squared of ka plus Nl squared of ka.

So that's your full cross section. If k is very small, ka much less than 1, that's small k. That's small energy, long wavelength approximation. This formula is interesting for high energy, for low energy, for intermediate energies. The angular dependence is interesting. It's a lots of things you could look at. Let's look at low energy.

You need the expansions of this quantities for small things, and they're easy to find. Basically, this remain finite. This becomes infinite. It dominates it. It's not difficult. I'll write what 1 gets. It's 4 pi over k squared, sum over l equals 0 to infinity, 1 over 2 l plus 1 2 to the l l factorial 2l factorial. So a mess of factorials. I'm sorry. ka to the 4l plus 2.

And here is to the question of convergence if ka is much smaller than 1, in this case-- this was the approximation-- that will sure converge. The powers go up and up. So for l equals 0, it is interesting, is the dominant one. So l equals 0 only sigma turns out to be 4 pi over k squared. And all these factors just give you a 1 for l equals 0. The ka squared, a squared, ka squared, which is 4 pi a squared, which is a pretty cute answer.

This is the low frequency cross section. If it's a long wave length approximation, the cross section of the object is not the apparent cross section, which is the diameter, but it's actually the full area of the sphere 4 pi a squared. People try to imagine why this is. It's almost like the wavelength is so big that the wave wraps around the sphere and gets stuck there. So the sphere captures proportional to the area of the whole thing. So we'll see more of that next time.