

PROFESSOR: OK. Let's do another application. I leave one thing. So this is going to be-- but it's called Landau levels. And they're pretty interesting. So it's the problem of solving for the motion of an electron in a magnetic field.

So it's called Landau levels for the physicist Lev Landau from Russia that discovered or did this calculation first. So you have a mass m charge q , and the magnetic field b in the z direction. We will solve it. And I think you will find it pretty interesting.

Well, you will be left with a little of an uneasy feeling, I think, at the end of the lecture, because the problem of gauging variance is so dramatic that the physics will look a little strange and a bit unrecognizable.

So let me remind you that in classical physics, if you have a magnetic field, b , you can have electrons that perform circular orbits, and the main property of those orbits is that they all run at the same frequency called cyclotron frequency, $q\hbar/mc$.

So this is just Newton's law, and the force being v over c cross b should be a couple of lines for you to remember that particle in the magnetic field goes in circles with that angular frequency.

So OK, there's lots of comments we could make about this, but let's assume we're going to solve for the motion of this particle. We're going to ignore spin. Sometimes spin is important. And when you have spin, you have a Zeeman effect, spin in a magnetic field. So that can change the energy levels by some quantities.

Let's ignore it for the moment. It is sometimes relevant. Sometimes it's not relevant, and you can easily take care of it. So again, we're faced with having to represent the magnetic field. So you have the magnetic field [INAUDIBLE] of a , and we'll take a solution in which a this time is minus by 0 and 0 .

Remember, that b is dx ay minus dy ax . And therefore this works out. It has an ax component, the derivative with respect to y gives you the magnetic field b . So what happens to your Hamiltonian?

Your Hamiltonian is now $\frac{1}{2m} p_x^2$ minus $\frac{q}{c}$ times ax plus $\frac{1}{2m} p_y^2$ plus $\frac{1}{2m} p_z^2$. I will not consider z motion. It's too simple. It's not too interesting. It's

just motion, plane waves in the z direction. It's not interesting. All the physics is really happening in the plane. It's the idea that we can constrain ourselves to have orbits of electrons that go like this in circles.

OK. So our Hamiltonian is here. Let's write it once again, $2m p_x$ plus $q b$ over c squared plus 1 over $2m p_y$ squared. In order to solve a system like that, it's convenient to see what is conserved, and p_x is conserved. p_x commutes with the Hamiltonian, because the Hamiltonian has no x dependence.

So h commutes with p_x . It doesn't commute with p_y , because there is a y in there. And, OK, so if it commutes with p_x , we can hope for eigenstates, energy eigenstates that are also eigenstates of p_x . So I will write my wave function of x and y as a wave function that depends on y times e to the $i k_x x$.

And it's already a little strange. We're looking for circular orbits maybe, and the x dependence is really a little funny here. It's almost like plane waves in the x direction. OK, well, it's a fact. It's true. You could not do that if you would have chosen a more symmetric gauge. You see, if you had chosen a more symmetric gauge, in which a doesn't have just an x component, but a y component. The y component would have depended on x , and then x would-- p_x and p_y , neither one would have commuted with the Hamiltonian.

So things would have been different. But this gauge is easy to solve equations. It's called a Landau gauge. And let's just explore what this says. Trust that if you're solving the equations, and you get a legal solution, it must mean the right thing.

So here, in this solution here, p_x is just $\hbar k_x$, and though we can restrict the action of the Hamiltonian to that subspace of momentum, $p_x = \hbar k_x$, we can look at the Hamiltonian, it is like looking at the Hamiltonian for this wave function, we already constrain ourselves to have this kind of momentum.

So the Hamiltonian that acts on the y function is the Hamiltonian constrained to the momentum k_x , and it's equal to 1 over $2m p_y$ squared, the second term, plus 1 over $2m q b$ over $c y$ plus $\hbar k_x$. It's the $q b$ over c plus the value of p_x , which is $\hbar k_x$, and I should square it.

If you look at it carefully, you see that this is a simple harmonic oscillator, Hamiltonian. There's p_y squared, and there's a y where the origin shifted. It's not just a y squared, but it's some sort of $y - y_0$ at some point squared.

So let's write it like that. So I have to make it look like a harmonic oscillator, $\frac{p^2}{2m} + \frac{1}{2}m\omega^2 y^2$. For a harmonic oscillator, I should have an m here. So I'm going to have an m squared in the denominator, so m . Let's get the $q\hbar$ mc squared out. y , I got that out, minus-- I have to put all these things. Minus $\hbar k x$ over-- times the c over $q\hbar$.

Look at all that. So we factor out $q\hbar$ over c . With the m squared here, and the m -- this is the 1 over m there. y , and I put two minus signs, because I always like to write y minus some y_0 , which is the equilibrium position of the oscillator.

Here, as you pull the c and the $q\hbar$ out, you get all these things. So this is a $p^2 y$ over $2n$, plus 1 over $m q\hbar$ over mc^2 squared y minus y_0 squared with y_0 equal minus $\hbar k x c$ over $q\hbar$.

That's it. That's the system. One good thing happened already. While this thing is a little funny or strange, I think, at first sight, this harmonic oscillator is resonating with an ω , which is the cyclotron frequency. So that's nice. The harmonic oscillator is that.

And therefore, already, you know you're going to get levels that are going to be separated by $\hbar \omega$ cyclotron. So that classical cyclotron frequency is going to separate the levels, and these levels are called the Landau levels, the various Landau levels.

OK, so let's see what this means, or how it looks like. For that, let's imagine a solution. So how does a solution look? So here is x and y . And suppose I take a kx , which is negative. This kx is any number you wish at this moment.

So kx negative means y_0 positive. So here is the y_0 is here. And the wave function is certainly the probability densities independent of x . So the wave function sort of has support all over x here.

And it represents an oscillator in the y direction. In the y direction, you're oscillating, and well, you're going to oscillate a little. We know, in particular, is a length scale associated with the oscillator. So let's look at that length scale.

The length scale on an oscillator \hbar over $m \omega$, in general. But now, we have an expression for ω . So \hbar over m , and ω is $q\hbar$ over mc . So this is square root of hc over $q\hbar$. And we're going to call this the magnetic length. So this is for an oscillator, an arbitrary oscillator takes these values. So let's call the magnetic length hc over $q\hbar$. This

magnetic length.

So that's a nice definition. And that's roughly the width of the state. Imagine you're in the ground state, you oscillate from the harmonic oscillator, the typical length scale, and in this case, is the magnetic length. So in the ground state, this size is l_b . And that's how your orbit looks. In particular, y_0 has a simple interpretation here.

y_0 is equal to minus k_x times the square of this magnetic length is a $[\text{? right ?}]$ squared. So it's minus k_x times l_b squared. And the units work out because k has units of 1 over length, and you have a length squared. So that's y_0 .

So that's how the orbits look. Now, you would say OK, so where are the circular orbits? Well, they're not quite so visible here. You have to do some work to find them. And in particular, what is happening here is a very strange thing.

I think-- well, I'll say one more thing. If you have this as your harmonic oscillator, your energies, that may depend of k_x and n_y , those are the quantum numbers you have already. Well, this is just a harmonic oscillator. So it's \hbar , the cyclotron frequency. Some people call the Landau frequency. Occupation number for the oscillator times $1/2$, the usual formula for a harmonic oscillator.

And here is the energy. And well, you had the plane wave. Why don't we have the p squared over $2m$ of a plane wave? It's nowhere to be found. That doesn't show and doesn't contribute to the energy. The mathematics was shown for us. So this kind of dependence doesn't do anything. It's the absolute degeneracy, the k_x .

So you could construct the superposition of states here, using Fourier transforms in which you somehow localize this in the x direction, by superimposing degenerate energy eigenstate. So this is the most important thing about this Landau levels. The Landau levels are the different levels of n_y , so n_y equals 0 is the lowest Landau level. Then you go n_y equal 1, n_y equal 2, n_y equal 3. But each Landau level is infinitely degenerate because you can put different values of k_x , and the energy doesn't change.