

PROFESSOR: We have a problem. The general perturbation theory, again, end states. But this time, the degeneracy is not broken. So degeneracy not lifted at first order.

Surprisingly, this subject is, as you can see, you have to do things with care. It's not in any of the textbooks that I know. In fact, I don't know of any place where it is discussed in detail or discussed anywhere, even though this result is pretty useful and you need it, and I think people improvise when they do this.

Even this discussion that with it here is not in any of your textbooks. In [INAUDIBLE], the advanced quantum mechanics, there's some discussion and mentions this factor and how subtle it is, but doesn't quite do everything till the bitter end. This case is harder. This is simple compared to what we're going to do now.

But we're going to sketch it so that you see the picture. We're going to do everything, but we're going to try to keep the picture clear because it gets sophisticated. So what's the situation that is happening. Physically, you have λ here, and you have the state.

And to first order, they don't split. So there are four states. And what we will assume is that the first order, they don't split. So they sort of come together.

And then to second order, they will split. We will assume that the second order, they will split. If they don't split the second order, things get complicated.

I'm actually curious about that. But I don't know of any physics example that they don't split to first nor to second order. You will have in the homework things that don't split to first order. That really happens often.

Then to second order, if you have four states and then split, means that the second order they'll go like this. And they'll start looking like that maybe, something like that. So the slope is the same to begin with, but then they split.

OK. So degeneracy not lifted at first order. So what is the problem here that the degeneracy is not lifted to first order? The problem is physically that this concept of a good basis, that basis that corresponds to the continuous changing eigenstates. You did the perturbation theory to first order, and you still don't know what is the good basis. Because the good basis is the one that the states change continuously.

So suppose that we had an example where we've had in two dimensions the conventional eigenstates, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, but the good basis was the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. And you would say, oh, they jumped. But you have to choose the right basis.

You do first order to generate perturbation theory. And if the states split, you have the good basis. But if they don't split, you have no idea what is the good basis. So in that sense, you're trying to go high, because we did first order. Now, you have to go to second order of perturbation theory to see if you can find what the good basis is.

So we will have to diagonalize something to second order to find the good basis. And once you have the good basis, you sort of have to start from the beginning. Now, I have a good basis. Let's do the perturbation theory nicely.

So if the degeneracy not broken, we have that $\langle n_0k | \delta H | n_0l \rangle$ is δ_{kl} . It is diagonal still. You do have a basis that diagonalizes the perturbation. You still diagonalize it. But once you diagonalize it, all the eigenvalues are the same. There's no account for m_l here. All of them are the same.

So what is the catch? The catch is that you now need to find good eigenstates. So what is the good basis? We'll write it in the following way. I will form linear combinations $k = 1$ to n of the $|n_0k\rangle$. And I will put coefficients a_k here.

And I will say, look, if you give me some coefficients a_k , I know how to superimpose now the N states. And let's hope that that's one good basis vector. You know, I've written one good basis vector if a_k it's a column vector of numbers. It gives me what is the superposition of those degenerate states that makes one vector. So I'll call this a vector $|\psi_0\rangle$.

So I think of this a_k not as unknown numbers, unknown vector that specifies a good state. And moreover, if I do things right, I should calculate these things. And I should find that my equations end up giving me N of those good vectors.

You see, if I specify a column vector of constants here, I get one vector here, a linear superposition. But I should get n linear superpositions. Because at the end of the day, there has to be n good vectors in that basis.

So in some sense, I hope for an a_0 with an index k , with k going from 1 to n , in which this by varying k provides me n column vectors, each one of which gives me a state. So this gives

me one state. If I had N of those coefficients, I would get n state. So that's my hope.

So suppose we get one of those good states, how should it look in perturbation theory? Well, the state Ψ_λ is going to be the state Ψ_0 . That's, in fact, the zeroth order state. We don't know those coefficients. That's our ignorance about the zeroth order states now.

$\Psi_0 + \lambda \Psi_1 + \dots$. And the energy's E_λ is E_0 , the 0 energy, plus the first order energy correction plus the second order energy corrections. And from this, the Schrodinger equation that as usual says that $H \Psi_\lambda = E_\lambda \Psi_\lambda$ gives you these equations that, by now, you're pretty familiar.

And we'll need a couple of them. So I will write them again. To order λ^0 , you get $H_0 \Psi_0 - E_0 \Psi_0 = 0$. That's a trivial equation because Ψ_0 is made up of states, all of which have energy E_0 . So that's always a trivial equation.

Then to order λ^1 , you have $H_0 \Psi_1 - E_0 \Psi_1 + H_1 \Psi_0 - E_1 \Psi_0 = 0$. We've done this already a few times. So I'm sure you don't have trouble believing this.

And to order λ^2 , the last one that I'll write, $H_0 \Psi_2 - E_0 \Psi_2 + H_1 \Psi_1 - E_1 \Psi_1 + H_2 \Psi_0 - E_2 \Psi_0 = 0$. OK. Kind of tired of writing these equations.

OK. So let's see what we should do. So we can start calculating things. And the way I'm going to do it, for the benefit of time, is that I will indicate the steps and what you do and skip the algebra. It will all be in the notes.

And in fact, the whole calculation will, if you want to go to the very end of it and exploit it to a maximum, there's lots of things one can do. And some of that will be left for homework. So our idea here is to give you the road map and for you to see how we're doing things.

So let me see here. We can do a few things with these equations. So the first equation is trivial, so we don't bother with it.

And the second equation, it's kind of interesting to see what it does. We can find from this equation a confirmation of what first order perturbation theory does. If we inject here one of the states in V_N , those N_0 's, you get the statement, again, that ΔH should diagonalize the perturbation. So I think I'll skip that. We'll discuss it in the notes.

But let's do the case when this equation gives something non-trivial. So let's calculate p acting

on the order λ equation. So when we had N^0 , say, on the order λ equation, that is kind of familiar. We've already done it a few times, so I'll skip it.

But let's do the p acting on this equation. So $p^0 H^0 \psi^0 - E^0 \psi^0 = p^0 H^1 \psi^0 - E^1 \psi^0 + \dots$. H^0 is a known Hamiltonian. And this we know the energy.

So this gives you $E^0 - E^0$ on $p^0 \psi^0 = 0$. Here is a state in \hat{V} . And this is a state in the degenerate subspace. We wrote it up there. That's our answer. We don't know even the zeroth order states. That's our problem.

But this term is orthogonal to that. So across the number, it gives us 0. On the other hand, here what do we get? We get minus the sum. Think of this state being the sum that we wrote in the previous thing. The sum from $k=1$ to n $\Delta H_{pk} a_k^0$. I've substituted this formula for the state ψ^0 in there.

And now, we have computed these overlaps because this is in principle known in terms of a^0 . And now we have these constants, these overlaps of the states outside the general space with the first correction and this part. So I'll write it this way. ψ^1 of \hat{V} is equal to $p^1 \psi^0$.

And what is this p^1 ? It's those coefficients we're trying to find here. So this is the component of ψ^1 along ψ^0 . So that component should go here.

And therefore, it's -1 over $E^0 - E^0$ all. And here we have sum from $k=1$ to n $\Delta H_{pk} a_k^0$.

OK. So we're pretty close to our goal, believe it or not. So what have we achieved? Let's see, again, where we are.

We have no idea what are the good states, so we introduced a^0 . a^0 parameterizes our ignorance of the good states. So we know, however, that the first order correction doesn't lift the degeneracy. And therefore, I don't have to recalculate that from the second equation. I just find the piece of ψ^1 that is also determined from that equation. And now, we've done completely the first order equation.

And what did we learn? We learned that the first order correction to the state is known if I know the good basis. Because all this first order correction to the state in the outside subspace in the rest is determined if I know the a^0 's. But I don't know the a^0 's. So we are calculating things, but we have not solved our problem.

But we run out of equations with the order λ . We've finished. No more λ equation, so we have to look into this equation now.