PROFESSOR: Today we have to continue with our discussion of the hydrogen atom. We had derived or explained how you would derive the corrections to the original Hamiltonian, the Hamiltonian you've studied already in a couple of courses, this h0 Hamiltonian for the hydrogen atom that has a kinetic term and a potential term.

This Hamilton we've studied for a while. And what we've said was that the Dirac equation provided a way to get systematically all the corrections, the first order of that Hamiltonian, and that's what we have here. That was the end product of that discussion in which the better Hamiltonian or the perturbed Hamiltonian including more effects was the original one plus a relativistic correction because this is not the relativistic kinetic energy.

Then there was a spin orbit coupling in which you couple the spin of the electron to the orbital angular momentum of the vector. And finally, there is a Darwin term. It's a surprising term. This is Charles Gaston Darwin, a British physicist, that discovered this term.

And all these terms were suppressed with respect to h0, in the sense that h0 had energies. The energies associated to h0 went like alpha squared mc squared. The fine structure constant, that's for h0 veera. But for h1 or for delta h, the energies go like alpha to the fourth mc squared, where m is the mass of the electron, and that's about 20,000, 19,000 times smaller.

So this is our perturbation, and that's what you have to understand. So we'll begin with the Darwin term. Then we'll turn to the relativistic term, then to the spin orbit term, and, by the end of today's lecture, we'll put it all together. So that's our plan.

So let's start with the Darwin term. What is this term? So Darwin term, I will try to evaluate it. So it depends on the potential energy. It has the Laplacian of the potential energy. So the potential energy is minus e squared over r.

So how much is the Laplacian of the potential energy? Well, the Laplacian of v would be minus e squared times the Leplacian of 1 over r. And that's minus e squared times minus 4 pi delta function of r. That's something you study in in E and M. The Laplacian of 1 over r is related to a delta function. It's the charge density of a point particle that produces that potential. So our result here is that this Laplacian is e square 4 pi e squared delta of r. It's a delta function contribution.

So let's write it here. The delta h Darwin is pi over 2 e squared h squared over m squared c squared delta of r. All right, so that's our correction. And we want to do perturbation theory with it, how it affects the energy levels of the various states of hydrogen. So how are they changed? What does it do to them?

Now, there is a simplifying fact here, the delta function. So you remember, first order corrections are always obtained by taking the interactive Hamiltonian and putting states in it. And the first thing we notice is that unless the wave function of the states does not vanish at 0, the correction vanishes. So this will pick up the value of the wave functions.

When you have two states, arbitrary state 1 and state 2 and delta h Darwin here-- there's two wave functions that you're going to put here if you're trying to compute matrix elements of the perturbation. And if either of these wave functions vanishes at the origin, that's not possible. But all wave functions vanish at the origin, unless the orbital angular momentum l is equal to 0.

Remember, all wave functions go like r to the l near the origin. So for l equals 0, that goes like 1, a constant, and you get a possibility. So this only affects l equals 0 states. That's a great simplicity.

Not only does that, but that's another simplification. Because at any energy level, l equals 0 states is just one of them. If you consider spin, there are two of them. Remember in our table of hydrogen atoms states go like that. So here are the l equals 0 states. And those are the only ones that matter.

So the problem is really very simple. We don't have to consider the fact that there are other degenerate states here. We just need to focus on l equals 0 states because both states have to be l equals 0 and l equals 0.

So our correction that we can call e1 Darwin for n 0 0-- because l is equal to 0, and therefore m is equal to 0-- would be equal to psi n zero zero delta h Darwin psi n 0 0.

So what is this? We have this expression for delta h Darwin. So let's put it here, pi over 2 e squared h squared over m squared c squared. And this will pick up the values of the wave functions. This overlap means integral over all of space of this wave function complex conjugated, this wave function and delta h Darwin. So at the end of the day, due the delta function, it gives us just psi n 0 0 at the origin squared. That's all it is. Simple enough.

Now, finding this number is not that easy. Because while the wave function here for l equals 0 is simple for the ground state, it already involves more and more complicated polynomials over here. And the value of the wave function at the origin requires that you normalize the wave function correctly.

So if you have the wave function and you have not normalized it properly, how are you going to get the value of the wave function at the origin?

This whole perturbation theory, we always assume we have an orthonormal basis. And indeed, if you change the normalization-- if the normalization was irrelevant here, this number would change with the normalization, and the correction would change. So you really have to be normalized for this to make sense.

And finding this normalization is complicated. You could for a few problems maybe look look up tables and see the normalized wave function, what it is. But in general here, there is a method that can be used to find this normalization analytically. And it's something you will explore in the homework.

So in the homework there is a way to do this, a very clever way, in which it actually turns out. So in the homework, you will see that the wave function at the origin for l equals 0 problems is proportional to the expectation value of the derivative of the potential with respect to r. So it's something you will do.

This potential, of course, is the 1 over r potential in our case. And the derivative means that you need to evaluate the expectation value of 1 over r squared in this state. But the expectation values of 1 over r squared in the hydrogen atom is something you've already done in the previous homework. It's not that difficult.

So the end result is that this term is calculable. And psi of n 0 0 at the origin squared is actually 1 over pi n cube a0 cube.

With that in, one can substitute this value and get the Darwin correction. You can see there is no big calculation to be done. But you can rewrite it in terms of the fine structure constant Darwin. And it's equal to alpha to the fourth mc squared 1 over 2n cubed. And it's valid for l equals 0 states.

So this number goes in here, and then write things in terms of the fine structure constant, and

then out comes this resolved.